HOMEOMORPHIC MEASURES IN METRIC SPACES

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ABSTRACT. For any nonatomic, normalized Borel measure μ in a complete separable metric space X there exists a homeomorphism $h: \mathfrak{N} \to X$ such that $\mu = \lambda h^{-1}$ on the domain of μ, where $\mathfrak{N}$ is the set of irrational numbers in (0, 1) and λ denotes Lebesgue-Borel measure in $\mathfrak{N}$. A Borel measure in $\mathfrak{N}$ is topologically equivalent to λ if and only if it is nonatomic, normalized, and positive for relatively open subsets.

1. Definitions and results. A *topological measure space* is a pair (X, μ), where X is a topological space and μ is a measure on the class of Borel subsets of X. (X, μ) is *homeomorphic* to (Y, ν) if there exists a homeomorphism of X onto Y that makes ν correspond to μ, and then ν is said to be *topologically equivalent* to μ. If B is a Borel subset of (X, μ), then μ_B denotes the restriction of μ to the class of Borel subsets of B. A measure μ is *everywhere positive* if $\mu(G) > 0$ for every nonempty open set G, *nonatomic* if $\mu(\{x\}) = 0$ for each $x \in X$, and *normalized* if $\mu(X) = 1$.

Let $\mathfrak{N}$ denote the set of irrational numbers in $I = [0, 1]$, and let λ denote the restriction of Lebesgue measure m to the Borel subsets of $\mathfrak{N}$. It is known [**8**, Theorem 2, p. 886] that a Borel measure in the n-dimensional cube I^n is topologically equivalent to n-dimensional Lebesgue-Borel measure in I^n if and only if it is everywhere positive, nonatomic, normalized, and vanishes on the boundary. A similar theorem will be shown to hold in $\mathfrak{N}$.

THEOREM 1. *A topological measure space* (X, μ) *is homeomorphic to* $(\mathfrak{N}, \lambda)$ *if and only if* X *is homeomorphic to* $\mathfrak{N}$ *and* μ *is an everywhere positive, nonatomic, normalized Borel measure in* X. *In particular, any such measure in* $\mathfrak{N}$ *is topologically equivalent to* λ.

It is known [**2**, §6, Exercise 8c, p. 84] that if X is a compact metric space, and μ is a nonatomic, normalized Borel measure in X, then (X, μ) is *almost homeomorphic* to (I, λ), in the sense that there exist sets $A \subset I$ and $B \subset X$ such that $\lambda(I - A) = 0$, $\mu(X - B) = 0$, and (B, μ_B) is homeomorphic to (A, λ_A). We shall show that this conclusion still holds when X is a complete separable metric space, and that the set A can always be taken equal to $\mathfrak{N}$.

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THEOREM 2. *If X is a topologically complete separable metric space, and μ is a nonatomic, normalized Borel measure in X, then there exists a G_δ set B in X such that $\mu(X-B)=0$ and (B, μ_B) is homeomorphic to $(\mathfrak{N}, \lambda)$.*

Any uncountable complete separable metric space X contains a copy of $\mathfrak{N}$ [6, Corollary 2, p. 352]. Theorem 2 implies that the most general nonatomic, normalized Borel measure in such a space can be constructed by mapping $\mathfrak{N}$ into X by a homeomorphism h, and then defining $\mu(E)=\lambda h^{-1}(E)$ for every Borel set E. The completion of μ is equal to mh^{-1}.

Theorem 2 can be generalized immediately to nonseparable spaces whose separability character has measure zero; such a space has an open set of measure zero whose complement is separable [7, Theorem III, p. 137]. On the other hand, the indispensability of completeness and metrizability is indicated by the following remarks.

REMARK 1. A separable metric space with a nonatomic, normalized Borel measure need not contain a copy of $\mathfrak{N}$.

Let X be a subset of I such that both X and $I-X$ meet every nonempty perfect subset of I. Then X has outer Lebesgue measure one and inner measure zero. Any relatively Borel subset A of X is of the form $A=X\cap B$, for some Borel set B in I. The formula $\mu(A)=m(B)$ defines unambiguously a nonatomic, normalized Borel measure μ in the separable metric space X, but X contains no copy of $\mathfrak{N}$.

REMARK 2. A compact Hausdorff space with a nonatomic, normalized, regular Borel measure need not contain a copy of $\mathfrak{N}$, or even of the set $\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$.

The Stone space X corresponding to any finite nonatomic measure algebra admits a nonatomic, normalized, regular Borel measure [5, §24]. Since X is compact and basically disconnected, every infinite closed subset contains a copy of βN [4, Problem 9H.2, p. 137], and so its cardinal is at least 2^c. The product of uncountably many copies of (I, m) is another example in which every compact metrizable subspace has measure zero [2, §8, Exercise 14a, p. 110].

2. **Proofs of Theorems 1 and 2.** A metrizable space is homeomorphic to $\mathfrak{N}$ if and only if it is topologically complete, separable, 0-dimensional, and nowhere locally compact [1, Satz IV, p. 95]. Hence any nonempty open subset of $\mathfrak{N}$ is homeomorphic to $\mathfrak{N}$. Likewise, any G_δ set that is both dense and frontier in some topologically complete, separable, 0-dimensional space Y is homeomorphic to $\mathfrak{N}$ [6, Theorem 3, p. 349].

LEMMA 1. *If μ is an everywhere positive, nonatomic, finite Borel measure in $\mathfrak{N}$, and if $\{\alpha_i\}$ is a sequence of positive real numbers such that $\sum_1^\infty \alpha_i = \mu(\mathfrak{N})$, then there exists a partition of $\mathfrak{N}$ into open sets U_i such that $\mu(U_i) = \alpha_i$ for all $i \in \boldsymbol{N}$.*

PROOF. Let $a(i, j) = j\alpha_i/(j+1)$ $(i \in \boldsymbol{N}, j \in \boldsymbol{N})$. Let $\{r_k\}$ $(k \in \boldsymbol{N})$ be an increasing sequence of rational numbers in $(0, 1)$, and let $r_0 = 0$. Denote the interval $(r_{k-1}, r_k) \cap \mathfrak{N}$ by $I(i, j)$, where (i, j) is the kth term in the ordering of $\boldsymbol{N} \times \boldsymbol{N}$ defined by $(i, j) < (i', j')$ if and only if $i+j < i'+j'$, or $i+j = i'+j'$ and $j < j'$. We wish to determine the sequence $\{r_k\}$ in such a way that

$$a(i, j) < \sum_{n=1}^{j} \mu(I(i, n)) < a(i, j+1)$$

for all i and j. Using the fact that $\mu([0, x] \cap \mathfrak{N})$ is a strictly increasing continuous map of $[0, 1]$ onto $[0, \mu(\mathfrak{N})]$, it is easy to see that such a sequence $\{r_k\}$ can be defined inductively, and that r_k will necessarily tend to 1. Then the sets $U_i = \bigcup_{j=1}^\infty I(i, j)$ constitute a partition of $\mathfrak{N}$ with the required properties.

LEMMA 2. *If μ and ν are two everywhere positive, nonatomic, Borel measures in $\mathfrak{N}$, if ρ is a metric compatible with the topology of $\mathfrak{N}$, and if U and V are open sets such that $\mu(U) = \nu(V) > 0$, then for each $\epsilon > 0$ there exist partitions $\{U_i\}$ of U and $\{V_i\}$ of V into nonempty open sets of diameter less than ϵ such that $\mu(U_i) = \nu(V_i)$ for all $i \in \boldsymbol{N}$.*

PROOF. Let $\{H_i\}$ $(i \in \boldsymbol{N})$ be a partition of V into nonempty open sets of diameter less than ϵ. Since U is a copy of $\mathfrak{N}$, by Lemma 1 there exists a partition $\{G_i\}$ of U into open sets such that $\mu(G_i) = \nu(H_i)$ for all i. Let $\{G_{ij}\}$ $(j \in \boldsymbol{N})$ be a partition of G_i into nonempty open sets of diameter less than ϵ. Since H_i is a copy of $\mathfrak{N}$ there exists a partition $\{H_{ij}\}$ of H_i into open sets such that $\mu(G_{ij}) = \nu(H_{ij})$ for all j. The families $\{G_{ij}\}$ and $\{H_{ij}\}$ constitute partitions of U and V having the required properties.

PROOF OF THEOREM 1. It suffices to prove the second assertion. Let ρ be a metric with respect to which $\mathfrak{N}$ is complete. By repeated application of Lemma 2 we obtain partitions $\boldsymbol{U}_n = \{U(i_1, \cdots, i_n)\}$ and $\boldsymbol{V}_n = \{V(i_1, \cdots, i_n)\}$ of $\mathfrak{N}$ into nonempty open sets of ρ-diameter less than $1/n$ such that $U(i_1, \cdots, i_n) \supset U(i_1, \cdots, i_{n+1})$, $V(i_1, \cdots, i_n) \supset V(i_1, \cdots, i_{n+1})$, and $\mu(U(i_1, \cdots, i_n)) = \lambda(V(i_1, \cdots, i_n))$ for all $n \in \boldsymbol{N}$ and all sets of indices $i_k \in \boldsymbol{N}$. For each $x \in \mathfrak{N}$ there is a unique sequence $\{i_n\}$ such that $x \in \bigcap_{n=1}^\infty U(i_1, \cdots, i_n)$. Define $T(x) = \bigcap_{n=1}^\infty V(i_1, \cdots, i_n)$. Then T is a 1-1 map of $\mathfrak{N}$ onto itself, and

$T(U(i_1, \cdots, i_n)) = V(i_1, \cdots, i_n)$. Since the union of each of the families $\{U_n\}$ and $\{V_n\}$ is a base with the property that any open set is the union of some disjoint subclass, it follows that T is a homeomorphism and that $\mu(E) = \lambda(T(E))$ for every Borel set E in $\mathfrak{N}$.

PROOF OF THEOREM 2. Let $\{x_i\}$ be a countable dense sequence in X. Let $\{r_j\}$ be a sequence of positive real numbers tending to zero such that $\mu(\{x:\rho(x, x_i) = r_j\}) = 0$ for all i and j. (Such a sequence exists because for each i, $\mu(\{x: \rho(x, x_i) = r\}) > 0$ for at most countably many values of r.) Let $S_{ij} = \{x: \rho(x, x_i) = r_j\}$ and $U_{ij} = \{x: \rho(x, x_i) < r_j\}$. Then $\{U_{ij}\}$ is a base for the topology of X. Let S be the union of all the sets S_{ij}, and let G be the union of all the sets U_{ij} such that $\mu(U_{ij}) = 0$. Then $\mu(G \cup S) = 0$, the G_δ set $Y = X - (G \cup S)$ is a topologically complete, separable, 0-dimensional subspace, and μ_Y is everywhere positive. Let D be a countable dense subset of Y, and let $A = G \cup S \cup D$. Then $\mu(A) = 0$, and the G_δ set $B = Y - D = X - A$ is both dense and frontier in Y. Hence B is homeomorphic to $\mathfrak{N}$, and μ_B is an everywhere positive, nonatomic, normalized Borel measure in B. By Theorem 1, (B, μ_B) is homeomorphic to $(\mathfrak{N}, \lambda)$.

3. **Uniqueness of** $(\mathfrak{N}, \lambda)$. Let $\mathfrak{F}$ denote the class of topological measure spaces (X, μ), where X is metrizable, separable, and topologically complete (i.e. a Polish space), and μ is a nonatomic, normalized Borel measure in X. The following theorem shows that $(\mathfrak{N}, \lambda)$ is the only member of this class that is topologically contained in each member of the class.

THEOREM 3. *If* (X, μ) *is a member of* $\mathfrak{F}$ *that is homeomorphic to a subspace of each member of* $\mathfrak{F}$, *then* (X, μ) *is homeomorphic to* $(\mathfrak{N}, \lambda)$.

PROOF. By hypothesis, (X, μ) is homeomorphic to some subspace (Y, λ_Y) of $(\mathfrak{N}, \lambda)$. Since $\lambda(Y) = 1$, Y must be a dense subset of $\mathfrak{N}$. It follows that Y is nowhere locally compact, as well as topologically complete, separable, and 0-dimensional. Consequently Y, and therefore X, is homeomorphic to $\mathfrak{N}$. Since λ_Y is everywhere positive, μ must also be. Therefore (X, μ) is homeomorphic to $(\mathfrak{N}, \lambda)$, by Theorem 1.

4. **Approximation of a Borel set by a Cantor subset.** As an application of Theorem 2 we give a new proof of the following theorem, recently proved by Gelbaum [**3**]. As the referee has pointed out, this theorem is implicitly contained in a result of von Neumann [**9**, Hilfsatz, p. 577].

THEOREM 4. *Let* X *be a complete separable metric space, and let* μ *be* *a nonatomic Borel measure in* X. *Any Borel set* A *in* X *with* $0<\mu(A)<\infty$ *is the union of a Cantor set and a set of arbitrarily small measure.*

PROOF. The formula $\nu(E)=\mu(E\cap A)/\mu(A)$ defines a nonatomic, normalized Borel measure ν in X. By Theorem 2, there exists a set B in X such that $\nu(X-B)=0$ and (B, ν_B) is homeomorphic to $(\mathfrak{N}, \lambda)$, say by h. Then $h(A\cap B)$ is a Borel subset of $\mathfrak{N}$ with $\lambda(h(A\cap B))=1$. Let C be a compact perfect subset of $h(A\cap B)$ with $\lambda(C)>1-\epsilon$. Then $h^{-1}(C)\subset A$, $\mu(A-h^{-1}(C))<\epsilon\mu(A)$, and $h^{-1}(C)$ is compact, perfect, and 0-dimensional, therefore homeomorphic to the Cantor set.

NOTE ADDED IN PROOF. A result closely related to Theorem 2 is contained in the recent paper by W. Böge, K. Krickeberg, and F. Papangelou, *Über die dem Lebesgueschen Mass isomorphen topologischen Massräume*, Manuscripta Math. **1** (1969), 59–77, Corallar, p. 72.

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