PROPERTIES OF ABSOLUTE SUMMABILITY MATRICES

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1. **Introduction.** Let A denote a matrix summability method that maps the complex number sequence x into the sequence Ax whose nth term is given by

$$(Ax)_n = \sum_{k \geqq 1} a_{nk} x_k.$$

If Ax is in l^1 whenever x is in l^1, then A is called an l-l matrix. An l-l matrix A is said to be sum-preserving if for each x in l^1,

$$\sum_{n \geqq 1} (Ax)_n = \sum_{k \geqq 1} x_k.$$

The inverse image of l^1 under A is denoted by l_A.

In [7, p. 129] Steinhaus proved that if A is a regular (i.e., limit-preserving) matrix, then there is a sequence of 0's and 1's such that Ax is not convergent. It follows immediately that a regular matrix cannot sum every bounded sequence. (Cf. [6].) The principal result of this paper is the analogue of this theorem for l-l matrices.

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2. **The main theorem.**

THEOREM 1. *If A is a sum-preserving l-l matrix and $p>1$, then $l^p \not\subseteq l_A$.*

By using the characterization of sum-preserving l-l matrices given in [5], we see that this theorem is an immediate corollary to the following assertion.

THEOREM 2. *If $p>1$ and A is an l-l matrix such that*

$$\limsup_k \sum_{n \geqq 1} | a_{nk} | > 0, \tag{1}$$

then $l^p \not\subseteq l_A$.

PROOF. We may assume that for each n, $\lim_k a_{nk} = 0$, for otherwise the conclusion is trivial. Let ϵ be a positive number such that for infinitely many k, $\sum_{n \geqq 1} |a_{nk}| \geqq 2\epsilon$. We now construct integer sequences κ and ν in the following manner.

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Choose $\kappa(1)$ so that $\sum_{n\geqq 1}|a_{n,\kappa(1)}|\geqq 2\epsilon$, then choose $\nu(1)$ such that

$$\sum_{n=1}^{\nu(1)}|a_{n,\kappa(1)}|>\epsilon \quad\text{and}\quad \sum_{n>\nu(1)}|a_{n,\kappa(1)}|<2^{-1}.$$

Having defined $\kappa(i)$ and $\nu(i)$ for $i<m$, we choose $\kappa(m)$ greater than $\kappa(m-1)$ such that

$$\sum_{n=1}^{\nu(m-1)}|a_{n,\kappa(m)}|<2^{-m} \quad\text{and}\quad \sum_{n\geqq 1}|a_{n,\kappa(m)}|\geqq 2\epsilon.$$

Then choose $\nu(m)$ greater than $\nu(m-1)$ and satisfying

$$\sum_{n=1+\nu(m-1)}^{\nu(m)}|a_{n,\kappa(m)}|>\epsilon \quad\text{and}\quad \sum_{n>\nu(m)}|a_{n,\kappa(m)}|<2^{-m}.$$

If $k=\kappa(i)$ define $x_k=i^{-1}$, otherwise $x_k=0$; obviously x is in l^p. Defining $\nu(0)=0$, we have

$$\begin{aligned}\sum_{n=1}^{\nu(N)}|(Ax)_n| &= \sum_{m=1}^{N}\sum_{n=1+\nu(m-1)}^{\nu(m)}\left|\sum_{i\geqq 1} i^{-1}a_{n,\kappa(i)}\right|\\ &\geqq \sum_{m=1}^{N}\sum_{n=1+\nu(m-1)}^{\nu(m)}\left\{|a_{n,\kappa(m)}|\,m^{-1}-\sum_{i\neq m}|a_{n,\kappa(i)}|\right\}\\ &> \sum_{m=1}^{N}\epsilon m^{-1}-\sum_{m=1}^{N}\sum_{n=1+\nu(m-1)}^{\nu(m)}\sum_{i\neq m}|a_{n,\kappa(i)}|.\end{aligned}$$

Since

$$\sum_{m\geqq 1}\sum_{n=1+\nu(m-1)}^{\nu(m)}\sum_{i<m}|a_{n,\kappa(i)}|<\sum_{m\geqq 1}2^{-m},$$

and

$$\sum_{m\geqq 1}\sum_{n=1+\nu(m-1)}^{\nu(m)}\sum_{i>m}|a_{n,\kappa(i)}|<\sum_{m\geqq 1}2^{-m-1},$$

it follows that Ax is not in l^1.

3. **Further remarks and results about l_A.** It is worthwhile noting that Theorem 2 gives a necessary condition for A to map l^p into l^1: viz.,

$$\text{(2)}\qquad \lim_k \sum_{n\geqq 1}|a_{nk}| = 0.$$

However, this condition is not sufficient to imply that $l^p \subseteq l_A$, even for diagonal matrices; e.g., consider diag $\{1/\text{Log}(n+1)\}$.

Property (2) does yield some information about the summability field l_A, but rather than limiting the size of l_A, (2) implies that l_A cannot be too small. This statement is put in precise language in the next result, which is easily proved.

PROPOSITION. *If A is a matrix such that*

$$\liminf_k \sum_{n\geq 1} |a_{nk}| = 0, \tag{3}$$

then $l_A \not\subseteq l^p$.

By an obvious modification of the proof of Theorem 2 we can show that a sum-preserving *l-l* matrix cannot map every sequence of 0's and 1's into l^1. This is precisely the *l-l* analogue of Steinhaus' theorem. In [2] R. C. Buck used the Steinhaus theorem to prove a summability characterization of convergent sequences: viz., the bounded sequence x is convergent if and only if there exists a regular matrix that sums every subsequence of x. In the light of the preceding remarks one might conjecture that members of l^1 could be characterized by the existence of a sum-preserving *l-l* matrix for which l_A contains every subsequence. This, however, is false.

EXAMPLE. Let A be the matrix mapping given by

$$(Ax)_1 = x_1 - x_2 \quad \text{and} \quad (Ax)_n = 2x_n - x_{2n-1} - x_{2n} \qquad \text{if } n > 1.$$

Then A is clearly a sum-preserving *l-l* matrix, but if x is a constant sequence then $(Ay)_n \equiv 0$ for every subsequence y of x.

REFERENCES

1. R. P. Agnew, *Summability of subsequences*, Bull. Amer. Math. Soc. **50** (1944), 596–598. MR **6**, 46.

2. R. C. Buck, *A note on subsequences*, Bull. Amer. Math. Soc. **49** (1943), 898–899. MR **5**, 117.

3. J. A. Fridy, *A note on absolute summability*, Proc. Amer. Math. Soc. **20** (1969), 285–286. MR **38** #1428.

4. G. H. Hardy, *Divergent series*, Clarendon Press, Oxford, 1949, MR **11**, 25.

5. K. Knopp and G. G. Lorentz, *Beiträge zur absoluten Limitierung*, Arch. Math. **2** (1949), 10–16. MR **11**, 346.

6. I. Schur, *Über linear Transformationen in der Theorie die unendlich Reihen*, J. Reine Angew Math. **151** (1921), 79–111.

7. H. Steinhaus, *Some remarks on the generalization of limit*, Prace Mat. Fiz. **22** (1911), 121–134.

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