INJECTIVE HULLS OF C^* ALGEBRAS. II

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Introduction. The aim of this paper is to continue the study of the category of commutative C^* algebras as initiated in [2]. Several results have already been referred to in the epilogue of [2]. Others include a new construction of the injective hull, a comparison with the category of Boolean algebras, and remarks on projective objects.

1. A new construction of the injective hull. We show how an explicit representation for injective hulls can be obtained for the C^* algebra of all complex valued continuous functions on a compact Hausdorff space. This is done without finding the projective cover of the space. We use the fact that the injective objects are precisely the AW^* algebras.

The following lemma is found in [1] on p. 178.

LEMMA 1. *The Boolean algebra B of regular open sets of a compact Hausdorff space is isomorphic to the Boolean algebra C of Borel sets modulo sets of first category.*

THEOREM 1. *The injective hull of the algebra $C(X)$ of all complex-valued continuous functions on the compact Hausdorff space X is the algebra $B(X)$ of all bounded Borel functions modulo sets of first category.*

PROOF. The idempotents correspond to the Borel sets modulo sets of first category. Since, in addition, the idempotents generate $B(X)$, $B(X)$ is an AW^* and hence an injective algebra. The natural map U of $C(X)$ into $B(X)$ induced by the inclusion map is clearly a homomorphism. It is one-one since continuous functions which are not identically equal must differ on a set of second category. To complete the proof, it suffices by Theorem 6 in [2] to show that for every nonzero idempotent e in $B(X)$ there is an element $0 \neq f \in C(X)$ such that $e(Uf) = Uf$. Since e is represented by a characteristic function on a regular open set, this follows immediately from Urysohn's lemma. Q.E.D.

REMARK. By avoiding the maximal ideal spaces this theorem brings the subject closer to classical analysis. The relationship with the

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construction in [1] can be seen from the fact that the maximal ideal space of an injective algebra is the same as the Stone space of the Boolean algebra of its idempotents.

The algebra $A(X)$ of all bounded functions on X is an injective algebra containing $C(X)$. Thus we have the following corollary which does not mention injectivity:

COROLLARY. *There exists an embedding of $B(X)$ into $A(X)$ which is the identity on $C(X)$. Furthermore the image of the embedding is a retract of $A(X)$.*

This may be regarded as a sort of lifting theorem. To see this let T be an arbitrary such embedding and let e be an idempotent in $B(X)$ represented by the characteristic function on an open set U. $T(e)$ is idempotent. If $x \in \overline{U}$, $\exists f \in C(X)$ such that $f(U)=0$ and $f(x)=1$. Hence $ef=0$. Therefore, $T(e)f=0$ [we are identifying $C(X)$ with its image in $B(X)$]. Thus $T(e)(x)=0$. Hence $T(e)$ is identically 0 on $\overline{U}'$. Similarly, by considering $1-e$ we obtain that $T(e)$ is identically 1 on U. Thus $\{x: T(x)=1\}$ differs from U by a set of first category. It follows by taking limits that T maps every function in $B(X)$ into one of its representatives in $A(X)$.

II. **Special examples of injective hulls.** We first get a classification of all spaces which are not essential covers of smaller spaces.

THEOREM 2. *An infinite space is not an essential cover of a proper quotient space iff it is a one point compactification of a discrete space.*

PROOF. Let $X \cup \{\infty\}$ be a one point compactification of a discrete space and

$$X \cup \{\infty\} \xrightarrow{f} Y$$

be minimal onto. $x \in X \rightarrow \{x\}$ is open, hence $X \cup \{\infty\} - \{x\}$ is closed. Thus $f[X \cup \{\infty\} - \{x\}]$ is not onto. Therefore $y \neq x \Rightarrow f(y) \neq f(x)$. Since this is true for all $x \in X$, f is one-one. Hence f is not a proper cover.

Conversely suppose X has no proper essential quotient space. In particular the space $\overline{X}$ obtained by identifying y and z for arbitrary distinct points $y, z \in X$ is not an essential quotient space. Thus there exists a proper closed subset A of X such that $A \rightarrow X \rightarrow \overline{X}$ is onto. A can only be $X-\{y\}$ or $X-\{z\}$. Therefore either $\{y\}$ or $\{z\}$ is open. Since y and z are arbitrary, at most, one point in the space is not open, and since X is infinite, at least one point in the space is not open. If ω is the point such that $\{\omega\}$ is not open then by compactness any open set containing ω has a finite complement. This proves the result. Q.E.D.

We thus have a categorical characterization of one point compactifications of discrete spaces. Note that the projective cover of such a space is the Stone-Cech compactification of the same space. Using the ideas in [3] it is easily seen that such a space is a quotient space of every other space with the same projective cover. Finally we obtain the following:

COROLLARY. *An infinite dimensional C^* algebra is not an essential extension of a proper subalgebra iff it is the algebra of all bounded functions on a set which is generated by the constant functions and the characteristic functions on single points.*

Suppose A and B are two C^* algebras such that $A \subset B$ and B is injective. The injective hull of A in B is not necessarily unique *as a subalgebra of B* although it is, of course, unique up to isomorphism. The nonuniqueness occurs already in a very simple case.

THEOREM 3. *Let X be the space $\{0\} \cup \{1/n\}$ where n runs through all positive integers with its usual topology. Turn X into a measure space by letting the measure of each point be one. Then there are 2^c injective hulls of $C(X)$ in $L^\infty(X)$.*

Note. It is clear from Theorem 7 in [2] that $L^\infty(X)$ itself is not the injective hull of $C(X)$. Incidentally, by the discussion in [2, Theorem 3] can be translated into a theorem on operator rings on Hilbert space.

We require the following lemma:

LEMMA. *Let A and B be two C^* algebras such that $A \subset B$ and let I be a complement of A in B. Then the maximal essential extensions of A in B are precisely the complements of I in B which contain A.*

PROOF. The fact that complements of I are maximal essential extensions of A is the statement of Theorem 5 in [2]. Conversely, suppose $\bar{A}$ is a maximal essential extension of A. Now the natural map $A \to \bar{A} \to B \to B/I$ is monic since $A \cap I = 0$. Hence $\bar{A} \to B \to B/I$ is monic by definition of essential extension. Hence $\bar{A} \cap I = 0$. Clearly I is maximal with this property since any larger ideal intersects A properly, hence a fortiori it intersects $\bar{A}$ properly. Thus I is a complement of $\bar{A}$. Let $\bar{\bar{A}}$ be a complement of I containing $\bar{A}$. Then $\bar{\bar{A}}$ is a maximal essential extension of $\bar{A}$, hence a maximal essential extension of A. Since $\bar{A}$ is a maximal essential extension of A then necessarily $\bar{\bar{A}} = \bar{A}$. Hence $\bar{A}$ is a complement of I.

We now prove the theorem. Note first that the set of functions I which vanish outside $\{0\}$ is the unique complement of $C(X)$ in $L^\infty(X)$. Clearly $C(X)\cap I=0$. On the other hand if $f\notin I(\exists n)(f(1/n)\neq 0)$. Let $g(1/n)=1/f(1/n)$ and $g(x)=0$ otherwise. Then $0\neq fg\in C(X)$. This shows simultaneously that I is a complement and that there are no other complements.

We must find all the complements of I. First, fix a free ultrafilter F on $\{1/n\}$. Then to any bounded function f on $\{1/n\}$ there corresponds a number a uniquely determined by the property that $f^{-1}([x: a-\epsilon<x<a+\epsilon])\in F$ for all $\epsilon>0$. Extend f to a function $\bar{f}$ on x by letting $\bar{f}(0)=a$. Since $f\rightarrow f(0)$ is a continuous homomorphism, the set of all $\bar{f}$ obtained is a C^* subalgebra of $L^\infty(X)$. Denote this algebra by $C(F)$. If a continuous function on X is restricted to $\{1/n\}$ and then extended to X by the above process then the original function is obtained. This is so because F is a free ultrafilter. Therefore $C(X)\subset C(F)$. For future reference note that if F is a point ultrafilter then there exist continuous functions which are not fixed by this process.

If $f(1/n)=0$ for all n then $\bar{f}(0)=0$. Thus $C(F)\cap I=0$. Clearly $C(F)+I=L^\infty(X)$. Hence $C(F)$ is a complement of I.

Conversely suppose D is a complement of I. Since $L^\infty(X)$ is injective then necessarily $D+I=L^\infty(X)$. (This follows from the fact that D is injective and that $D\rightarrow L^\infty(X)\rightarrow L^\infty(X)/I$ is an essential extension. This implies that $D\rightarrow L^\infty(X)\rightarrow L^\infty(X)/I$ is an isomorphism.) It now follows from the definition of I that for every bounded function f on $\{1/n\}$ there is a unique function $\bar{f}$ in D whose restriction to $\{1/n\}$ is F. Consider the map $f\rightarrow\bar{f}(0)$. Since D is an algebra the map is a homomorphism. It is a standard result in Banach algebra theory that all such homomorphisms arise from ultrafilters. Hence D has the form $C(F)$. By an earlier remark F is necessarily free if D contains $C(X)$.

Finally note that there are 2^c ultrafilters on a countable set. This proves the theorem. Q.E.D.

III. **The relationship with the category of Boolean algebras.** Consider the full subcategory of the category of compact Hausdorff spaces whose objects are the totally disconnected spaces. Using Stone duality, the category of Boolean algebras may be regarded as a full subcategory of the category of C^* algebras. The identification may be expressed as follows: The Boolean algebra corresponding to a given C^* algebra is the Boolean algebra formed by its idempotents.

THEOREM 4. *The full subcategory of totally disconnected spaces is a reflective subcategory of the category of compact Hausdorff spaces with* *the reflection given by the passage from a compact Hausdorff space to its quotient by the equivalence relation*: $x \sim y$ *iff every clopen set containing* x *also contains* y.

The proof is left to the reader. Note that the reflective nature of the subcategory mentioned in the above theorem can be obtained as a special case of Theorem C in [5]. However, our result gives a simple explicit construction for the reflector.

This can also be done by means of the dual category. We can easily show that the full subcategory of algebras generated by idempotents is a coreflective subcategory of the category of C^* algebras. In fact if C is a C^* algebra, the coreflector $T(C)$ of C is the subalgebra generated by the idempotents. The proof that this is a coreflector is left to the reader. Thus the categories of Boolean algebras and C^* algebras are related in a manner which has lately become fashionable.

Many constructions such as the free product with amalgamation can be done in the category of C^* algebras in a manner which is similar to the method used in the category of Boolean algebras. As a final remark, both categories have the same injectives.

IV. **Injective spaces.** Returning to the main results in [1] it is natural to inquire about injective spaces and injective hulls. First, the closed unit interval is an injective cogenerator in the category of compact Hausdorff spaces. It follows that there are sufficiently many injectives and that the injectives are precisely the retracts of products of closed unit intervals. In particular, injectives are connected and homotopically trivial. In the full subcategory of totally disconnected spaces, the injectives are precisely the retracts of dyadic spaces. Thus in contrast to projectives, injectives in the two categories are different except in trivial cases.

The study of injective hulls evaporates because of the following theorem.

THEOREM 5. *In the category of compact Hausdorff spaces proper essential extensions do not exist.*

PROOF. Let $A \subset X$ and let $x \in X$ and $x \notin A$. Let y be arbitrary in X and consider the identification space $\overline{X}$ obtained by identifying x and y. Then the natural map $A \rightarrow X \rightarrow \overline{X}$ is monic although $X \rightarrow \overline{X}$ is not. Q.E.D.

In particular, it follows that an object which is not injective can not have an injective hull.

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