CONTINUED FRACTIONS OVER AN INNER PRODUCT SPACE

F. A. ROACH

1. Introduction. In this paper we introduce the notion of a continued fraction with elements which are points of a real inner product space and give three convergence theorems. Throughout this paper we assume that S is a real inner product space and that u is a particular point of S with norm 1. If x is a point, the point $2((x, u))u-x$ is referred to as the conjugate of x with respect to u, and since we do not consider conjugation with respect to any other point, it will be denoted by $\bar{x}$. Notice that if x and y are points and c is a real number, then

$$\overline{x+y} = \bar{x} + \bar{y}, \qquad \overline{cx} = c\bar{x}, \qquad \overline{(\bar{x})} = x, \quad \text{and} \quad \|\bar{x}\| = \|x\|.$$

If x is a point distinct from 0, the symbol $1/x$ will be used to denote the point $\bar{x}/\|x\|^2$. We assume there is adjoined to S a "point at infinity" ∞ with the usual conventions; $1/0=\infty$, $1/\infty=0$, etc. Notice that if T is a two dimensional subspace of S which contains u, we may regard it as being the set of all complex numbers, with u corresponding to unity, since within T, the transformation $1/z$ is equivalent to the ordinary reciprocal transformation.

If, for $n=1, 2, 3, \cdots$, b_n is a point, we refer to the expression

$$\frac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}} \tag{1.1}$$

as a continued fraction over S. The symbol f_n will be used to denote the nth approximate of (1.1). That is to say, $f_1 = 1/b_1$, $f_2 = 1/(b_1+1/b_2)$, $f_3 = 1/[b_1+1/(b_2+1/b_3)]$, etc. Of course, we say that (1.1) converges if and only if $f_1, f_2, f_3, \cdots$ has a finite limit; when convergent, that limit is called the value of (1.1).

The following notation proves useful: for $n=1, 2, 3, \cdots$ and $k=1, 2, \cdots, n$,

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$$(1.2)\qquad \begin{aligned} &\mathfrak{A}_{-1}^{n} = 0, \qquad \mathfrak{A}_{0}^{n} = 1, \quad \text{and} \\ &\mathfrak{A}_{k}^{n} = \bar{b}_{n-k+1}\|\mathfrak{A}_{k-1}^{n}\|^{2} + \bar{\mathfrak{A}}_{k-1}^{n}\|\mathfrak{A}_{k-2}^{n}\|^{2}. \end{aligned}$$

It follows by induction that for $n=1, 2, 3, \cdots$,

$$(1.3)\qquad f_n = (\|\mathfrak{A}_{n-1}^{n}\|^{2}\mathfrak{A}_{n}^{n})/\|\mathfrak{A}_{n}^{n}\|^{2}.$$

2. **A necessary condition for convergence.** The following theorem, for the complex case—the case in which each partial denominator of (1.1) is a complex number—was given by von Koch [4]. For a proof and discussion of this result, see Wall [5, pp. 27–28].

THEOREM 1. *If, for* $n=1, 2, 3, \cdots$, b_n *is a point and the series* $\sum\|b_i\|$ *converges, then the continued fraction* (1.1) *diverges.*

We will first establish an identity involving the symbols (1.2). Namely, that for $n=1, 2, 3, \cdots$

$$(2.1)\qquad \begin{aligned} &\|(\|\mathfrak{A}_{n-1}^{n}\|^{2}\|\mathfrak{A}_{n+1}^{n+1}\|^{2})\mathfrak{A}_{n}^{n} - (\|\mathfrak{A}_{n}^{n+1}\|^{2}\|\mathfrak{A}_{n}^{n}\|^{2})\mathfrak{A}_{n+1}^{n+1}\| \\ &\qquad = \left(\prod_{i=1}^{n}\|\mathfrak{A}_{i}^{n}\|\right)\left(\prod_{i=1}^{n+1}\|\mathfrak{A}_{i}^{n+1}\|\right). \end{aligned}$$

Making use of this identity and the convergence of the series named above, we will show that the approximants of (1.1) do not form a Cauchy sequence.

Since S is an inner product space, the square of the left-hand side of (2.1) may be written as

$$\|\mathfrak{A}_{n+1}^{n+1}\|^{2}\|\mathfrak{A}_{n}^{n}\|^{2}[\|\mathfrak{A}_{n-1}^{n}\|^{4}\|\mathfrak{A}_{n+1}^{n+1}\|^{2} - 2\|\mathfrak{A}_{n}^{n+1}\|^{2}((\mathfrak{A}_{n}^{n}, \mathfrak{A}_{n+1}^{n+1})) + \|\mathfrak{A}_{n}^{n+1}\|^{4}\|\mathfrak{A}_{n}^{n}\|^{2}].$$

With this, we see that the left-hand side of (2.1) is

$$\|\mathfrak{A}_{n+1}^{n+1}\|\cdot\|\mathfrak{A}_{n}^{n}\|\cdot\|\ \|\mathfrak{A}_{n-1}^{n}\|^{2}\mathfrak{A}_{n+1}^{n+1} - \|\mathfrak{A}_{n}^{n+1}\|^{2}\mathfrak{A}_{n}^{n}\|,$$

which, with the aid of (1.2), may be written as

$$(2.2)\qquad \|\mathfrak{A}_{n+1}^{n+1}\|\cdot\|\mathfrak{A}_{n}^{n}\|\cdot\|\ \|\mathfrak{A}_{n-2}^{n}\|^{2}\|\mathfrak{A}_{n}^{n+1}\|^{2}\mathfrak{A}_{n-1}^{n} - \|\mathfrak{A}_{n-1}^{n+1}\|^{2}\|\mathfrak{A}_{n-1}^{n}\|^{2}\mathfrak{A}_{n}^{n+1}\|.$$

It is now easily seen that (2.1) holds true in case $n=1$. The truth of (2.1) follows at once by induction.

Dividing both sides of (2.1) by $\|\mathfrak{A}_{n+1}^{n+1}\|^{2}\|\mathfrak{A}^{n}\|^{2}$, we obtain

$$(2.3)\qquad \|f_n - f_{n+1}\| = \left(\prod_{i=1}^{n-1}\|\mathfrak{A}_{i}^{n-1}\|\right)\left(\prod_{i=1}^{n}\|\mathfrak{A}_{i}^{n}\|\right)\Big/(\|\mathfrak{A}_{n+1}^{n+1}\|\|\mathfrak{A}_{n}^{n}\|).$$

We now introduce some additional notation: for $n=1, 2, 3, \cdots$,

$$(2.4)\quad D_n = \|b_n\| \|b_{n-1} + 1/b_n\| \cdots \|b_1 + \underline{1}/\overline{b_2} + \cdots + \underline{1}/\overline{b_n}\|.$$

Since

$$\underline{1}/\overline{b_{n-k}} + \underline{1}/\overline{b_{n-k+1}} + \cdots + \underline{1}/\overline{b_n} = (\|\overset{n}{\mathcal{A}}_k\|^2 \overset{n}{\mathcal{A}}_{k+1})/\|\overset{n}{\mathcal{A}}_{k+1}\|,$$

by means of a simple calculation, the right-hand side of (2.3) may be reduced to $1/(D_n D_{n+1})$. Thus we have

$$(2.5)\qquad \|f_n - f_{n+1}\| = 1/(D_n D_{n+1}).$$

Clearly,

$$D_1 \leqq 1 + \|b_1\|,$$

$$D_2 \leqq \|(\|b_2\| b_1 + \bar{b}_2/\|b_2\|)\| \leqq (1 + \|b_1\|)(1 + \|b_2\|),$$

and, by induction, it follows that

$$(2.6)\qquad D_n \leqq (1 + \|b_1\|)(1 + \|b_2\|) \cdots (1 + \|b_n\|).$$

However, if the series $\sum\|b_i\|$ is convergent, then the product $\prod(1+\|b_i\|)$ is also convergent. Therefore, in view of (2.6), the sequence $D_1, D_2, D_3, \cdots$ does not increase without bound and so, with the aid of (2.5), we see that $f_1, f_2, f_3, \cdots$ does not form a Cauchy sequence. From this we conclude that the divergence of the series $\sum\|b_i\|$ is necessary for the convergence of the continued fraction (1.1).

REMARK 2.1. Notice that in the proof of the above theorem, we did not need to use directly the fact that $\bar{x} = 2((x, u))u - x$. In fact, if T is a linear transformation from S onto S of norm 1 such that T^2 is the identity transformation, we may interpret $\bar{x}$ to be $T(x)$ and this theorem remains true.

REMARK 2.2. One might raise the question as to whether the above theorem can be generalized in a manner similar to the generalization of von Koch's theorem due to Scott and Wall [1].

3. **A sufficient condition for convergence.** The following theorem, for the complex case, may be found in a paper by Thron [2]. In this case, if $c = 0$, the theorem reduces to a theorem due to Worpitzky [6]. For a discussion of this result, see Wall [5, pp. 42–44]. We use the term *convergence set* to mean a point set having the property that if b_n lies in it, $n = 1, 2, 3, \cdots$, the continued fraction (1.1) converges. If M is a point set by the *value region* associated with M is meant the set of all points which, for some sequence $b_1, b_2, b_3, \cdots$ from M, are either one of the approximants of (1.1) or else, in case (1.1) converges, the value of (1.1).

THEOREM 2. *Suppose that S is complete and c is a real number. If M is the set of all points z such that*

$$\|z - cu\| \geqq (c^2 + 4)^{1/2}, \tag{3.1}$$

then M is a convergence set. Moreover, the value region associated with M is the set of all points z such that

$$\|2z + cu\| \leqq (c^2 + 4)^{1/2}. \tag{3.2}$$

We will first establish that the value region is given by (3.2). To this end, let M_1 denote M and V_1 denote $1/M_1$ (i.e., V_1 is the set of all points z such that, for some point x of M_1, $z = 1/x$). Let M_2 denote the point set $M_1 + V_1$—the set of all points z such that for some point x of M_1 and some point y of V_1, $z = x + y$—and let V_2 denote $1/M_2$. Denote by M_3 the point set $M_1 + V_2$ and by V_3, the point set $1/M_3$, etc. Thus, for $n = 1, 2, 3, \cdots$, M_{n+1} is $M_1 + V_n$ and V_{n+1} is $1/M_{n+1}$. By a simple argument, we see that V_1 is the set of all points z such that

$$\|z - (1/a + 1/b)u/2\| \leqq (1/a - 1/b)/2$$

where $a = c + (c^2+4)^{1/2}$ and $b = c - (c^2+4)^{1/2}$. We also find that V_2 is the set of all points z such that

$$\|z - [1/(a + 1/b) + 1/(b + 1/a)]u/2\| \leqq [1/(a + 1/b) - 1/(b + 1/a)]/2,$$

etc. Since

$$\underline{1}/\overline{a} + \underline{1}/\overline{b} + \underline{1}/\overline{a} + \cdots$$

has value $-[c - \sqrt{(c^2+4)}]$ and

$$\underline{1}/\overline{b} + \underline{1}/\overline{a} + \underline{1}/\overline{b} + \cdots$$

value $-[c + \sqrt{(c^2+4)}]$, we see that V is the set of all points z such that (3.2) holds true.

Suppose that, for $n = 1, 2, 3, \cdots$, b_n lies in M and let K_n denote the point set

$$\underline{1}/\overline{b_1} + \underline{1}/\overline{b_2} + \cdots + \underline{1}/\overline{b_n + V}.$$

Now, for $n = 1, 2, 3, \cdots$, let b_n' denote the complex number x such that $|x| = \|b_n\|$, $\mathrm{Re}(x) = ((b_n, u))$, and $\mathrm{Im}(x) \geqq 0$. Denote by M' the set of all complex numbers z such that, with $u = 1$, (3.1) holds true and by V' the corresponding value region. Also, for $n = 1, 2, 3, \cdots$, let K_n' denote the set

$$\underline{1}/\overline{b_1'} + \underline{1}/\overline{b_2'} + \cdots + \underline{1}/\overline{b_n' + V'}.$$

It is not difficult to see that K_n is a sphere together with its interior, K_n' is a circle together with its interior, and that the radius of K_n is the same as the radius of K_n'. It is also easily seen that K_{n+1} is a subset of K_n and that K_{n+1}' is a subset of K_n'.

We will now show that the radius of K_n' tends to 0 as n increases. We have from [2] that

$$\underline{1}/\overline{b_1'} + \underline{1}/\overline{b_2'} + \underline{1}/\overline{b_3'} + \cdots$$

converges. Let v denote the value of this continued fraction and suppose that v' is a complex number which belongs to each one of K_1', K_2', K_3', $\cdots$ (Clearly v belongs to each of these sets.) Thus, we see that there exists a sequence $x_1, x_2, x_3, \cdots$ such that, for $n=1, 2, 3, \cdots$, x_n lies in V' and

$$v' = \underline{1}/\overline{b_1'} + \underline{1}/\overline{b_2'} + \cdots + \underline{1}/\overline{b_n' + x_n}.$$

There exists a complex number x and a subsequence s of $x_1, x_2, x_3, \cdots$ such that s converges to x. For $n=1, 2, 3, \cdots$, let

$$v_n = \underline{1}/\overline{b_1'} + \underline{1}/\overline{b_2'} + \cdots + \underline{1}/\overline{b_n' + x}.$$

Making use of the well-known recursion formulas for computing the value of v_n, we see that $v_1, v_2, v_3, \cdots$ has limit v. Notice that for $n=1, 2, 3, \cdots$, the continuous transformation

$$\underline{1}/\overline{b_1'} + \underline{1}/\overline{b_2'} + \cdots + \underline{1}/\overline{b_n' + z},$$

maps V' onto a subset of V'. It is not difficult to see that the subsequence of $v_1, v_2, v_3, \cdots$ which corresponds to s must converge to v'. Hence, $v=v'$ and from this we conclude that the radius of K_n tends to 0 as n increases. Therefore, if each one of $b_1, b_2, b_3, \cdots$ lies in M, the continued fraction (1.1) converges.

Remark 3.1. In view of the proof given above, one might suppose that all questions of convergence of continued fractions over S could be reduced to questions of convergence of ordinary continued fractions in this manner. However, this is not the case. Consider the following example in which $S=E_3$ and $u=(1, 0, 0)$. For $n=1, 2, 3, \cdots$, let b_{2n-1} denote $(0, 1, 0)$ and b_{2n} denote $(0, 0, 1)$. In this case, both b_{2n-1}' and b_{2n}' (as used above) would be the complex number i and the continued fraction

$$1/\overline{i} + 1/\overline{i} + 1/\overline{i} + \cdots$$

is divergent. But with these values for $b_1, b_2, b_3, \cdots$, (1.1) is convergent. This may be seen by observing that if, for $n=1, 2, 3, \cdots$, (x_n, y_n, z_n) denotes the nth approximant of (1.1), then $x_n=0$ and the complex number y_n+iz_n is the nth approximant of the periodic continued fraction

$$1/\overline{-1} + 1/\overline{i} + 1/\overline{-1} + \cdots$$

which is convergent. (See Wall [5, p. 37].)

REMARK 3.2. The region M of Theorem 2 is "best" in the sense that if r is a positive number which is less than $(c^2+4)^{1/2}$ and the right-hand side of (3.1) is replaced by r, the resulting statement does not hold true. This may be seen by simply considering the real case.

4. Van Vleck's theorem.

The theorem of this section, for the complex case, was first given by Van Vleck [3]. A proof differing from Van Vleck's and some extensions of this theorem due to Scott and Wall [1] are given by Wall [5, pp. 123–134].

THEOREM 3. *Suppose that S is complete, c is a positive number, and let M denote the set of all points z such that*

$$\|z - ((z, u))u\| \leqq c((z, u)).$$

If b_1 is a point such that $((b_1, u))>0$ and, for $n=1, 2, 3, \cdots$, b_n lies in M, then the continued fraction (1.1) *converges if and only if the series $\sum \|b_i\|$ diverges.*

If the series $\sum\|b_i\|$ converges, we have by Theorem 1 that the continued fraction (1.1) diverges. Hence, suppose that this series is divergent. Notice that since, for $n=1, 2, 3, \cdots$, b_n belongs to the "cone" M, the series $\sum((b_i, u))$ diverges also. This, together with the fact that $((b_1, u))>0$, implies that the continued fraction

$$1/\overline{((b_1, u))} + 1/\overline{((b_2, u))} + 1/\overline{((b_3, u))} + \cdots \tag{4.1}$$

converges since $((b_n, u)) \geqq 0$, $n=2, 3, 4, \cdots$. We will use this fact to establish that the approximants of (1.1) form a Cauchy sequence.

The point set M has the property that if z and z' belong to it, then so do $z+z'$, $\bar{z}$, and any nonnegative multiple of z. Therefore, if m and n are positive integers and $m>n$,

$$1/\overline{b_n} + 1/\overline{b_{n+1}} + \cdots + 1/\overline{b_m}$$

lies in M. Now, for $n=1, 2, 3, \cdots$, let K_n denote the point set

$$\frac{1}{|b_1} + \frac{1}{|b_2} + \cdots + \frac{1}{|b_n + M}.$$

Notice that for $m > n$, f_m lies in K_n. As in the proof of Theorem 2, we will show that d_n, the diameter of K_n tends to zero as n increases. Clearly, $d_1 \leqq 1/((b_1, u))$ since $\|b_1\| \geqq ((b_1, u))$. We also have, by means of a geometrical argument, that the point set $1/(b_2+M)$ is a subset of the spherical ball with center $u/2((b_2, u))$ and radius $1/2((b_2, u))$. From this it follows that

$$d_2 \leqq \left| 1/[((b_1, u)) + 1/((b_2, u))] - 1/((b_1, u)) \right| .$$

By a simple argument we conclude that, for $n = 2, 3, 4, \cdots$, d_n does not exceed the absolute value of the difference between the nth and the $(n+1)$st approximants of (4.1). However, (4.1) converges and hence we see that $f_1, f_2, f_3, \cdots$ is a Cauchy sequence.

References

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University of Georgia