COMMUTATIVE QF-1 ARTINIAN RINGS ARE QF

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Abstract. In a recent paper, D. R. Floyd proved several results on algebras, each of whose faithful representations is its own bicommutant (= R. M. Thrall's QF-1 algebras, a generalization of QF-algebras) among which was the theorem in the title for algebras. We obtain our extension of Floyd's result by use of interlacing modules, replacing his arguments involving the representations themselves.

In [10], Thrall observed that the class of finite-dimensional algebras over which every faithful representation has the double centralizer property (i.e., is its own bicommutant) properly contains the class of quasi-Frobenius (= QF) algebras. He called the members of the former class QF-1 algebras and posed the intriguing problem of characterizing these algebras in terms of ideal structure. Solutions for this problem have been given for generalized uniserial algebras [5] and for commutative algebras [4]. Every faithful module M over a QF ring R has the double centralizer property in the sense that the natural homomorphism \[ \lambda: R \to \text{Hom}_C(M, M) \] (where \( C = \text{Hom}_R(M, M) \)) is onto (see [2, §59]). Thus Thrall's definition and his problem extend naturally to QF-1 artinian rings.

In view of recent results on the dominant dimension of an artinian ring (see [1, Theorem 2] or [8, Lemma 9]), the characterization of QF-1 generalized uniserial algebras (and its proof) given in [5] remains valid for generalized uniserial rings. In this note we prove the theorem of the title, thus extending a theorem that Floyd proved for finite-dimensional algebras by means of matrix representations [4].

It is not difficult to show that a direct sum of rings is QF or QF-1 if and only if so is each of the direct summands. Thus for our purposes we may assume that R is a commutative local artinian ring. According to Nakayama's original definition [9, p. 8], such a ring is QF if and only if its R-socle (i.e., its largest semisimple R-submodule) \( S(R) = S(R) \) is simple. We shall prove the theorem by constructing, in the event that \( S(R) \) is not simple, a faithful module whose double centralizer has an R-socle larger than that of R. The methods used in this

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construction are suggested by the Lemma of [7] and Theorem 3.1 of [3].

Let $R$ be a commutative local artinian ring and let $M$ be a finitely generated indecomposable $R$-module with centralizer $C = \text{Hom}_R(M, M)$ and double centralizer $C' = \text{Hom}_C(M, M)$. Let $K = R/\text{Rad} R$ and $D = C/\text{Rad} C$. In this setting we have the following.

1. The ring $C$ is completely primary, in the sense that $D$ is a division ring and $\text{Rad} C$ is nilpotent [6, Chapter 4].

2. Since $R$ is commutative, $C$ and $C'$ are algebras over $R$, via

$$(r\gamma)(m) = r\gamma(m) = \gamma(rm) = (\gamma r)(m)$$

for $r \in R$, $\gamma \in \text{Hom}(M, M)$, $m \in M$.

3. $(\text{Rad} R) C$ is a nilpotent ideal in $C$, so $D$ is an algebra over the field $K$.

4. The $C$-socle of $M$, $S(cM)$, is an $R$-$C$-module annihilated by $\text{Rad} C$ and hence is a $K$-$D$-vector space.

With these observations we can now prove that the $R$-socle of $C'$, $S(rC')$, has length at least as large as the dimension of $S(cM)$ over $K$. That is,

$$|S(rC'):K| \geq |S(cM):K|.$$

**Proof.** Let $T$ be a maximal $C$-submodule of $M$. Then, because the functor $\text{Hom}_C(\ , \ )$ is left exact in both variables and $C$ is an $R$-algebra, there is an $R$-monomorphism

$$0 \rightarrow \text{Hom}_C(M/T, S(cM)) \rightarrow C'.$$

But since $\text{Rad} C$ (and hence $\text{Rad} R$) annihilates $M/T$ and $S(cM)$ this is really a $K$-monomorphism

$$0 \rightarrow \text{Hom}_D(M/T, S(cM)) \rightarrow S(rC').$$

Thus, since $M/T$ is a one-dimensional $D$-space and $S(cM)$ is a finite-dimensional $D$-space, we have

$$|S(rC'):K| \geq |\text{Hom}_D(M/T, S(cM)):K|$$


Now we are in a position to prove our main result.

**Theorem.** Every commutative artinian QF-1 ring is QF.

**Proof.** Suppose that $R$ is as above and has distinct (but necessarily isomorphic) minimal ideals $S$ and $S'$. Let $\phi: S \rightarrow S'$ be an isomorphism and form the interlacing module
$$M = (R \times R)/L, \quad L = \{(s, -\phi(s)) \mid s \in S\}.$$  

Then $M$ contains a copy of $R$ and so is faithful. Suppose $\gamma$ is an $R$-endomorphism of $M$. If $\eta: R \times R \to M$ is the natural epimorphism then, using the projectivity of $R \times R$, one obtains an $R$-map $\tilde{\gamma}$ making the diagram

$$\begin{array}{ccc}
R \times R & \xrightarrow{\tilde{\gamma}} & R \times R \\
\eta \downarrow & & \downarrow \eta \\
M & \xrightarrow{\gamma} & M
\end{array}$$

commute and consequently taking $L$ into $L$. The operation of $\tilde{\gamma}$ on $R \times R$ is just that of some matrix

$$\begin{pmatrix}
r_{11} & r_{12} \\
r_{21} & r_{22}
\end{pmatrix}, \quad r_{ij} \in R.$$  

The stability of $L$ under $\tilde{\gamma}$ yields for each $s \in S$, an $\tilde{s} \in S$ such that $s$ and $\tilde{s}$ satisfy the matrix equation

$$(s, -\phi(s)) \begin{pmatrix}
r_{11} & r_{12} \\
r_{21} & r_{22}
\end{pmatrix} = (\tilde{s}, -\phi(\tilde{s})).$$

From these equations and the independence of $S$ and $S'$ it follows easily that $r_{12}$ and $r_{21}$ both annihilate $S$ and so are nilpotent, and that $r_{11}$ and $r_{22}$ are simultaneously either nilpotent or invertible. Thus the matrix of $\tilde{\gamma}$ either has all nilpotent entries or is invertible. But $\gamma$ is nilpotent or invertible if $\tilde{\gamma}$ is, by the commutativity of the diagram, so $M$ is indecomposable. Moreover, if $\gamma$ is nilpotent then $\tilde{\gamma}$, having a matrix with radical entries, must annihilate $S(R) \times S(R)$. That is,

$$\gamma(\eta(S(R) \times S(R))) = \eta(\tilde{\gamma}(S(R) \times S(R))) = 0$$

whenever, in our earlier notation, $\gamma \in \text{Rad } C$. This proves that

$$\eta(S(R) \times S(R)) \subseteq S(cM)$$

and the containment is as $K$-spaces. Now surely

$$\left| \eta(S(R) \times S(R)):K \right| = 2 \left| S(R):K \right| - 1,$$

so by (5) the double centralizer $C'$ of $M$ must have an $R$-socle strictly larger than that of $R$. This completes the proof.

If $R$ is a semisimple ring then all $R$-modules (faithful or not) have the double centralizer property. Thus one wonders which rings satisfy this condition. Recalling that a ring is uniserial if and only if each of its factor rings is QF (see [5]), we obtain the
Corollary. Every module over a commutative artinian ring $R$ has the double centralizer property if and only if $R$ is a uniserial ring.

Added in proof. V. P. Camillo has independently obtained these results using different methods.

References


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