ZEROS OF ANALYTIC FUNCTIONS WITH INFINITELY DIFFERENTIABLE BOUNDARY VALUES

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Abstract. A necessary and sufficient condition is proved that a set of points \( \{ r_n e^{i \theta_n} \} \) in the unit disk be the set of zeros of an analytic function with infinitely differentiable boundary values for every choice of \( \{ r_n \} \), \( 0 < r_n < 1 \) and \( \sum (1 - r_n) < \infty \).

1. Introduction. The algebra \( A^\infty \) is the class of all functions analytic in the open unit disk \( D \) with all derivatives bounded in \( D \) or, alternatively, the class of all bounded analytic functions with boundary values \( f(e^{i \theta}) = \lim_{r \to 1} f(re^{i \theta}) \) having infinitely many continuous derivatives. Beurling [1, p. 13], Carleson [2], and Novinger [5] have characterized the boundary zeros of such functions, while Taylor and Williams [6] have discovered several further properties of this class relating to zeros. Little, however, is known about the zeros within \( D \) beyond a few partial results (see [4] and [8]).

In this paper, an apparently unrelated sufficient condition on the points is presented (Theorem 2). If \( z_n = r_n e^{i \theta_n} \) satisfy the Blaschke condition and if the closure of the set \( \{ e^{i \theta_n}; n = 1, 2, \ldots \} \) of projections of the points to the boundary \( T \) of \( D \) forms a Carleson set, then there is a nonzero function \( f \in A^\infty \) such that \( f(z_n) = 0 \). Thus the points may converge to their limit set as tangentially as desired provided they are "well spaced out."

Together with a slightly altered version of the construction in [4], Theorem 1, this result provides a necessary and sufficient condition that \( \{ r_n e^{i \theta_n} \} \) be the zeros of an \( A^\infty \) function for every choice of \( \{ r_n \} \), \( 0 < r_n < 1 \) and \( \sum (1 - r_n) < \infty \).

The construction requires some knowledge of \( A^\infty \) functions in other domains than the unit disk. This is discussed in §2, where the analogue of the Carleson-Novinger result on boundary zeros is formulated in a simply-connected Jordan domain with smooth boundary. §3 is devoted to some growth estimates used in the construction. Finally, the construction forms §4 of this paper.

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2. Boundary zeros of \( A^\omega \) functions. Let \( R \) be a domain in the complex plane, and let \( A^\omega(R) = \{ f : f^{(n)} \text{ is analytic and bounded in } R, \text{ for } n = 1, 2, \cdots \} \). In this section the possible boundary zeros of functions in \( A^\omega(R) \) are determined for a Jordan domain \( R \) with a smooth boundary, i.e., \( R \) is bounded by a rectifiable Jordan curve \( w = w(s) \), where \( w \) is infinitely differentiable with respect to arc length \( s \).

**Lemma 1.** If \( R \) is a Jordan domain with smooth boundary, then if \( \psi \) is the mapping function from \( R \) to the unit disk \( D \), \( \psi \in A^\omega(R) \) and \( \phi = \psi^{-1} \in A^\omega(D) \). Moreover, \( \phi'(z) \neq 0 \) for \( z \in \overline{D} \).

**Proof.** That \( \phi \in A^\omega(D) \) follows from a theorem of Kellogg (see [7]): if the boundary function \( w \) has \( n + 2 \) bounded derivatives with respect to arc-length (\( w \in C^1 \)), then \( \phi \) has \( n + 1 \) bounded derivatives in \( D \), and thus \( n \) derivatives continuously extendable to the closed disk \( \overline{D} \). Warschawski also showed \( \phi'(z) \) is nowhere zero in \( D \). Thus it is possible to solve for the derivatives of \( \psi \) in terms of those of \( \phi \), and therefore they are bounded in \( R \).

A Carleson set \( E \) is a closed set of measure zero contained in the unit circle \( T \) for which, if the intervals complementary to \( E \) have lengths \( \epsilon_n \), \( \sum \epsilon_n \log \epsilon_n > -\infty \). Novinger [5] showed that every Carleson set is the set of boundary zeros of an \( A^\omega \) function, while Beurling [1] proved that the zeros in \( T \) of any function, analytic in \( D \) and continuous in \( \overline{D} \) which satisfies a Lipschitz condition on \( T \), must form a Carleson set.

We define a Carleson set for a Jordan domain \( R \) with smooth boundary \( \partial R \) to be a closed set \( E \subseteq \partial R \) of linear measure zero, whose complementary arcs satisfy the same finiteness condition. The following lemma is an easy consequence of the boundedness of the first derivatives of the mapping functions.

**Lemma 2.** If \( R \) is a Jordan domain with smooth boundary, \( \psi \) is the mapping function from \( R \) to the unit disk, and \( E \subseteq \partial R \), then \( E \) is a Carleson set in \( \partial R \) if and only if \( \psi(E) \) is a Carleson set in \( T \).

**Theorem 1.** Let \( R \) be a Jordan domain with smooth boundary. If \( f \) is analytic in \( R \) and continuous in \( \overline{R} \), and if \( f \) satisfies a Lipschitz condition on \( \partial R \), then the zeros of \( f \) in \( \partial R \) form a Carleson set in \( \partial R \). Conversely, if \( E \) is a Carleson set in \( \partial R \), there is a function \( f \in A^\omega \) which has a zero of infinite order at each point of \( E \) (i.e., \( f^{(j)}(z) = 0, j = 0, 1, 2, \cdots \) for \( z \in E \), and no other zeros).

**Proof.** If \( f \) is analytic in \( R \) and satisfies the Lipschitz condition \( |f(z_1) - f(z_2)| \leq K|z_1 - z_2|^{\alpha} \) for \( z_1, z_2 \in \partial R \), then by Lemma 1, \( f \circ \phi \) is
analytic in $D$ and satisfies a Lipschitz condition of the same order, where $\phi$ is the mapping function from $D$ to $R$. By Beurling’s proof, the zero set of $f \circ \phi$ is a Carleson set, and by Lemma 2, the boundary zero set of $f$ is a Carleson set in $\partial R$.

If $E$ is a Carleson set in $\partial R$, Novinger’s construction provides a function $g \in A^\infty(D)$ vanishing on $\phi^{-1}(E)$ and nowhere else. By Lemma 1, $f = g \circ \phi^{-1}$ is the desired function.

3. Magnitude of Blaschke products and $A^\infty$ functions. In this section we present some estimates on the derivatives of Blaschke products and $A^\infty$ functions. Similar estimates were proved by Wells [8].

If $z_k \in D$ and $\sum (1 - |z_k|) < \infty$, then the Blaschke product with zeros $z_k$, 

$$B(z) = \prod \frac{\tilde{z}_k}{|z_k|} \frac{z_k - z}{1 - \tilde{z}_k z},$$

converges in $D$ to a bounded analytic function with radial limits $B(e^{i\theta})$ of modulus 1 almost everywhere. Any bounded function $f$ analytic in $D$ has a factorization $f = FB$, where $B$ is a Blaschke product and $F$ has no zeros. Thus estimates of the growth of the derivatives of Blaschke products are essential to the construction.

**Lemma 3.** If $B$ is a Blaschke product with zeros $z_k = r_ke^{i\theta_k}$, where $r_k > 1/2$, then there is a sequence of positive numbers $N_j$ for which, if $A$ is any subproduct of $B$,

$$|A^{(j)}(z)| \leq N_j \text{dist}(z, K)^{-2j}, \quad j = 1, 2, \ldots,$$

where $K = \{1/\tilde{z}_k: k = 1, 2, \ldots \}$.

**Proof.** Differentiating $B$,

$$B'(z) = \sum_{k=1}^\infty B_k(z) \frac{r_k^2 - 1}{(1 - \tilde{z}_k z)^2},$$

where $B_k(z) = B(z)(1 - \tilde{z}_k z)/(z_k - z)$. The modulus of $B'$ is thus bounded by

$$\sum_{k=1}^\infty \left| \frac{1 - r_k^2}{1 - \tilde{z}_k z} \right|^2 \leq \sum_{k=1}^\infty \frac{8(1 - r_k)}{|z_k^{-1} - z|^2} \leq 8 \text{dist}(z, K)^{-2} \sum_{k=1}^\infty (1 - r_k).$$

Since this estimate improves if a subproduct is taken in place of $B$, the lemma holds for $j = 1$. Suppose that constants $N_j$ have been determined for which the inequality holds for indices $j \leq m$. Then there exist positive numbers $a_{mj}$ such that
\begin{align*}
|A^{(m+1)}(z)| & \leq \sum_{k=1}^{\infty} (1 - r_k) \sum_{j=0}^{m} a_{mj} |A^{(m-j)}_k(z)| \left|1 - \bar{z}_k z\right|^{-(j+2)} \\
and
|A^{(m+1)}(z)| & \leq \sum_{k=1}^{\infty} (1 - r_k) \sum_{j=0}^{m} a_{mj} N_{m-j} \left|1 - \bar{z}_k z\right|^{-(j+2)} \text{dist}(z, K)^{-2(m-j)}
\end{align*}
by the inductive hypothesis. Since $\frac{1}{2} \text{dist}(z, K) \leq \left|1 - \bar{z}_k z\right| \leq 2$, this is bounded by
\begin{equation*}
\left(\sum_{k=1}^{\infty} (1 - r_k) \sum_{j=0}^{m} a_{mj} N_{m-j} 2^{j+2}\right) \text{dist}(z, K)^{-2(m+1)},
\end{equation*}
and the estimate holds for $A^{(m+1)}$.

**Lemma 4.** If $f \in A^\infty(R)$ and $f$ has a zero of infinite order at $z_0 \in \partial R$, and the line joining $z$ to $z_0$ lies in $R$, then there are positive constants $M_{jk}$ for which
\begin{equation*}
|f^{(j)}(z)| \leq M_{jk} |z - z_0|^k, \quad j, k = 0, 1, 2, \ldots.
\end{equation*}
This lemma is easily proved by integrating $f^{(k+j+1)}$ $k$ times.

**4. Construction of an $A^\infty$ function with given zeros.** In this section the required function is constructed after the construction of a curve which is fundamental to the argument.

**Lemma 5.** If $R_k e^{i\theta_k}$ is a sequence of points outside the unit disk with limit points in a closed set $E$ of zero Lebesgue measure in $T$, there is a rectifiable curve $r = h(\theta)$ which is $C^\infty$ with respect to arc-length, and for which $1 < h(\theta_k) \leq R_k$.

The proof is left to the reader.

**Theorem 2.** If $\{e^{i\theta_n}: n = 1, 2, \ldots\}$ is a set of points on the unit circle whose closure is a Carleson set, and if $0 < r_n < 1$ and $\sum (1 - r_n) < \infty$, there is a function $f \in A^\infty$ for which $f(r_n e^{i\theta_n}) = 0$. Conversely, if $\{r_n e^{i\theta_n}\}$ is contained in the zero set of an $A^\infty$ function for every choice of $\{r_n\}$, $0 < r_n < 1$ and $\sum (1 - r_n) < \infty$, then $\{e^{i\theta_n}\}$ is a Carleson set.

**Proof.** Let $z_n = r_n e^{i\theta_n} \in D$, where the closure of $\{e^{i\theta_n}: n = 1, 2, \ldots\}$ is a Carleson set and $\sum (1 - r_n) < \infty$. We may assume $r_n > 1/2$, for otherwise the function resulting from ignoring the points $z_n$ with $r_n \leq 1/2$ can be multiplied by the finite Blaschke product with those zeros.

By Lemma 5, there is a $C^\infty$ curve $r = h(\theta)$ between the unit circle and the points $R_n e^{i\theta_n}$, where $R_n = \inf \{1/r_k: \theta_k = \theta_n\}$, and $h(\theta) > 1$ unless $e^{i\theta}$ is a limit point of $\{z_n\}$. This curve will form the boundary of a
Jordan domain $R$. Let $K = \{1/\mathbb{Z}_n: n = 1, 2, \cdots \}$, and let $E$ be the closure of the set of points $h(\theta_n)e^{i\theta_n}$. It is clear that $E$ is a Carleson set in $\partial R$. By Theorem 1, there is a function $F \in A^\infty(R)$ which has a zero of infinite order at each point of $E$. Further, the restriction of $F$ to $D$ is in $A^\infty(D)$.

Let $B$ be the Blaschke product with zeros $z_n$, and let $f(z) = F(z)B(z)$ for $z \in D$. By Lemma 4, $|f^{(j)}(z)| \leq M_{jk} \text{dist}(z, E)^k \leq M_{jk} \text{dist}(z, K)^k$ for some constants $M_{jk}$ and $j, k = 0, 1, \cdots$. By Lemma 3 there are constants $N_j$ for which $|B^{(j)}(z)| \leq N_j \text{dist}(z, K)^{-2j}$. If $z \in D$,

$$|f^{(j)}(z)| = \left| \sum_{i=0}^{j} \binom{j}{i} F^{(j-i)}(z) B^{(i)}(z) \right| \leq \sum_{i=0}^{j} \binom{j}{i} M_{j-i,2i} N_i,$$

so $f^{(j)}$ is bounded, $j = 1, 2, \cdots$.

Conversely, if $\{e^{i\theta_n}\}$ is not a Carleson set, a slight modification of the construction in Theorem 1 of [4] yields a sequence $\{r_n\}$ for which $\{r_n e^{i\theta_n}\}$ is not contained in the zero set of any function with finite Dirichlet integral, and thus not of any $A^\infty$ function.

Necessary conditions are difficult to find. Carleson's formula for the Dirichlet integral [3] yields the condition

$$\int_0^{2\pi} \log \sum_n \frac{1 - |z_n|^2}{|e^{it} - z_n|^2} dt < \infty$$

which he used to create the counterexample in [4]. This, the Blaschke condition, and the requirement that the limit points lie in a Carleson set seem to be all that is known. The distance from these conditions to the known sufficient conditions is rather great.

References


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