OSCILLATION THEOREMS FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. The oscillatory and nonoscillatory behavior of the nonlinear second order differential equation (1) $x''+p(t)f(x)=0$ is related to that of $(2)_\lambda$ $x''+\lambda p(t)x=0$, $\lambda>0$. Under certain conditions on $p(t)$ and $f(x)$ it is shown that all solutions of (1) are oscillatory if $(2)_\lambda$ is oscillatory for all $\lambda>0$. In contrast to most of the literature on this subject, no sign or integrability conditions on $p(t)$ are explicitly assumed.

Consider the second order nonlinear differential equation

$$x'' + p(t)f(x) = 0 \tag{1}$$

where $p(t)\in C[0, +\infty)$ and $f(x)\in C^{(1)}(-\infty, +\infty)$, with

$$f'(x) \geqq \frac{f(x)}{x} > 0 \quad \text{for } x \neq 0. \tag{2}$$

As a special case we have

$$x'' + p(t)x^{2n+1} = 0. \tag{3}$$

In case $p(t)$ is eventually positive, oscillation and nonoscillation criteria for (1) and (3) have been extensively developed. (See [1] and the bibliography therein for the nonlinear case. Willett in [2] has an extensive bibliography for the case when (1) is linear.) However, much less is known for the nonlinear case when $p(t)$ is allowed to be negative for arbitrarily large values of t. It is the purpose of this paper to relate the oscillatory behavior of (1) with that of the linear equation

$$x'' + \lambda p(t)x = 0, \qquad \lambda > 0, \tag{$(4)_\lambda$}$$

which, presumably, is easier to handle. We shall restrict attention to solutions of (1) which exist on some ray $[T, +\infty)$ where T may depend on the particular solution. We shall at various times assume that the following condition holds:

$$\liminf_{t\to\infty} \int_T^t p(s)ds > 0 \quad \text{for all large } T. \tag{5}$$

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For the case when $f(x)=x^{2n+1}$, our main result generalizes a theorem due to Utz [4] who assumes $p(t)\geqq 0$ and a theorem due to Waltman [5] who has shown that all solutions of (3) oscillate provided the following condition holds:

$$\int^{\infty} p(s)ds = +\infty. \tag{6}$$

For $n=0$ we have by the well-known Fite-Wintner Theorem (see [6]) that condition (6) implies all solutions of (3) oscillate. In fact, we see that $(4)_\lambda$ is oscillatory for all $\lambda>0$.

LEMMA 1. *Let $u(t)$ be a nonoscillatory solution of* (1) *on* $[T, +\infty)$ *and let condition* (5) *hold. Then for all large t we have $u(t)u'(t)>0$.*

PROOF. Assume, to be specific, that $u(t)>0$ for $t\geqq T_1$, $T_1\geqq T$. Obvious modifications are valid when $u(t)<0$. If the lemma is not true, then either $u'(t)<0$ for all large t or $u'(t)$ oscillates. In the former case we may suppose that T_1 is sufficiently large so that

$$\int_{T_1}^{t} p(s)ds \geqq 0 \quad \text{for } t \geqq T_1$$

and $u'(t)<0$ for $t\geqq T_1$. Hence, we have

$$\begin{aligned}\int_{T_1}^{t} p(s)f(u(s))ds &= f(u(t))\int_{T_1}^{t} p(s)ds \\ &\quad - \int_{T_1}^{t} f'(u(s))u'(s)\int_{T_1}^{s} p(\sigma)d\sigma ds \geqq 0, \qquad t \geqq T_1.\end{aligned} \tag{7}$$

Now integrating (1) we have by (7) that $u'(t)\leqq u'(T_1)<0$, $t\geqq T_1$, which contradicts the fact that $u(t)$ is nonoscillatory.

If $u'(t)$ oscillates, let $T_n\to+\infty$ be such that $u'(T_n)=0$. For $t\geqq T_1$ we define

$$v(t) = -u'(t)/f(u(t)), \tag{8}$$

and differentiating, we get

$$v'(t) = p(t) + w(t), \tag{9}$$

where

$$w(t)=(v(t))^2 f'(u(t))\geqq 0, \qquad t\geqq T_1.$$

Since $v(T_n)=0$ we integrate (9) between T_n and T_{n+1} and sum on n to get an immediate contradiction to (5).

THEOREM 2. *Let* $(4)_\lambda$ *be oscillatory and let* $u(t)$ *be a nonoscillatory solution of* (1) *with* $u(t)u'(t)>0$ *for all* $t \geqq T$. *Then*

$$\lim_{t\to\infty} \frac{f(u(t))}{u(t)} \leqq \lambda. \tag{10}$$

PROOF. Let $g(x)=f(x)/x$. We note that condition (2) implies that the limit in (9) exists (possibly infinite). If the theorem is not true, we may assume $g(u(t)) \geqq \lambda$ for all $t \geqq T$. Let $z(t)$ be the solution of $(4)_\lambda$ satisfying $z(T)=0$, $z'(T)=1$, and let $T_1 > T$ be the first zero of $z'(t)$ so that $z'(t)>0$ on $[T, T_1)$. Then

$$\int_T^{T_1} (g(u(t)) - \lambda)(z')^2 dt \geqq 0 \tag{11}$$

so integrating by parts we get

$$\begin{aligned}
&\int_T^{T_1} (g(u(t)) - \lambda)(z')^2 dt \\
&\qquad = \lambda \int_T^{T_1} pz^2(g(u(t)) - \lambda)dt - \int_T^{T_1} zz'g'(u(t))u'dt \\
&\qquad \leqq \lambda \int_T^{T_1} pz^2(g(u(t)) - \lambda)dt
\end{aligned}$$

since the integrand $zz'g'(u(t))u'$ is nonnegative on $[T, T_1]$ by condition (2). But

$$\begin{aligned}
&\int_T^{T_1} pz^2(g(u(t)) - \lambda)dt \\
&\qquad = \int_T^{T_1} \frac{z}{u}(pzf(u(t)) - \lambda pzu)dt \\
&\qquad = \int_T^{T_1} \frac{z}{u}(z'u - u'z)'dt \\
&\qquad = -u'(T_1)(z(T_1))^2/u(T_1) - \int_T^{T_1} ((z'u - u'z)/u)^2 dt < 0,
\end{aligned}$$

and this is a contradiction.

Lemma 1 along with Theorem 2 imply the following:

COROLLARY 3. *Let* $(4)_\lambda$ *be oscillatory and assume condition* (5) *holds. Then all nonoscillatory solutions of* (3) *are bounded. In fact, if* $u(t)$ *is a nonoscillatory solution of* (3), *then*

$$\lim_{t\to\infty} |u(t)| = \gamma \leqq (\lambda)^{1/2n}.$$

THEOREM 4. *Assume* $(4)_\lambda$ *is oscillatory for all* $\lambda>0$ *and assume condition* (5) *holds. Then all solutions of* (1) *oscillate.*

PROOF. If not, let $u(t)$ be a nonoscillatory solution of (1). Lemma 1 and Theorem 2 imply $u(t)$ satisfies $u(t)u'(t)>0$ for all large t and $\lim_{t\to\infty} g(u(t))=0$. But this is a contradiction since $d(g(u(t)))/dt \geqq 0$ by (2). This proves the theorem.

Consider the following somewhat weaker condition than (5): There exists a sequence $T_n \to +\infty$ such that

$$\int_{T_n}^{t} p(s)ds \geqq 0, \qquad t \geqq T_n. \tag{5*}$$

The proof of Lemma 1 and Theorem 2 imply

COROLLARY 5. *If* $p(t)$ *satisfies condition* (5*), *and if* $(4)_\lambda$ *is oscillatory for all* $\lambda>0$, *then* $u'(t)$ *oscillates for all solutions* $u(t)$ *of* (1).

EXAMPLES. Willett [3] has shown that $(4)_\lambda$ is oscillatory for all $\lambda>0$ where $p(t)=t^\eta \sin t$ and $\eta>-1$. Thus, Corollary 5 implies that $u'(t)$ oscillates for all solutions $u(t)$ of (1) if $-1<\eta\leqq 0$.

For the equation

$$x'' + (\rho t^{-2} + \mu t^{-1} \sin \nu t)x = 0 \tag{12}$$

results in [3] imply oscillation if $\rho>\frac{1}{4}-\frac{1}{2}(\mu/\nu)^2$ and nonoscillation if $\rho<\frac{1}{4}-\frac{1}{2}(\mu/\nu)^2$. Moreover, $p(t)$ satisfies condition (5) if $\rho>\mu/\nu\geqq 0$. Letting $\mu=\nu$ and $\rho>1$ it follows that $x''+\lambda p(t)x=0$ is oscillatory if $\lambda>\lambda_0\equiv(\rho^2+\frac{1}{2})^{1/2}-\rho$ so that all nonoscillatory solutions of (3) satisfy $|u(t)|\leqq(\lambda_0)^{1/2n}$ for all large t by Corollary 3.

REFERENCES

1. James S. W. Wong, *On second order nonlinear oscillation*, Funkcial. Ekvac. **11** (1969), 207–234.

2. D. Willett, *Classification of second order linear differential equations with respect to oscillation*, Advances in Math. **3** (1969), 594–623.

3. ———, *On the oscillatory behavior of the solutions of second order linear differential equations*, Ann. Polon. Math. **21** (1969), 175–194.

4. W. R. Utz, *Properties of solutions of* $u''+g(t)u^{2n-1}=0$, Monatsh. Math. **66** (1962), 55–60. MR **25** #2275.

5. Paul Waltman, *An oscillation criterion for a nonlinear second order equation*, J. Math. Anal. Appl. **10** (1965), 439–441. MR **30** #3265.

6. A. Wintner, *A criterion of oscillatory stability*, Quart. Appl. Math. **7** (1949), 115–117. MR **10**, 456.

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