A DIRECT PROOF THAT A LINEARLY ORDERED SPACE IS HEREDITARILY COLLECTIONWISE NORMAL1

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Although it appears well known that a linearly ordered space is completely normal (=hereditarily normal), most available proofs (in, for instance, [1] and [2]) are very indirect. In this paper we present a direct proof of a stronger theorem, namely that the interval topology is hereditarily collectionwise normal.2

If X is linearly ordered, we will call a set S ⊆ X convex if a, b ∈ S and a < t < b implies t ∈ S. The union of any collection of convex sets with nonempty intersection is convex, so any subset S of X can be uniquely expressed as a union of disjoint maximal convex sets called convex components. Clearly every interval in X is convex but not conversely, and we will, as usual, denote intervals by (a, b), [a, b], [a, b), or [a, b]. In what follows, X will denote a linearly ordered space, i.e., a linearly ordered set endowed with the usual open interval topology.

Suppose {A_i} is a discrete family of subsets of X. Let

\[ A_i^* = \bigcup \{[a, b] \mid a, b \in A_i, [a, b] \cap A_j = \emptyset \ \forall j \neq i \}. \]

Then \[ A_i \subseteq A_i^* \] and \[ A_i^* \cap A_j^* = \emptyset \text{ whenever } i \neq j \]; in fact, the family \{A_i^*\} is discrete. To prove this, we select for each \( x \in X \) a neighborhood \( I_x \) which intersects at most one of the sets \( A_i \). If \( I_x \) meets exactly one element of \{A_i\}, say \( A_k \), and if \( x \) is not an endpoint of \( X \), we can take \( I_x \) to be an interval \((s, t)\). Then if \( i \neq k \), \((s, t)\) may intersect \( A_i^* \) only if it intersects some interval \([a, b]\) where \( a, b \in A_i \). But since \((s, t) \cap A_i = \emptyset \) and \( a, b \in A_i \), then \((s, t) \subseteq (a, b)\) which would imply that \( A_k \cap A_i^* = \emptyset \). But this is impossible if \( i \neq k \), so in this case \( I_x \) can intersect at most one of the sets \( A_i^* \). Other cases are treated analogously, so \{A_i^*\} (and consequently cl(A_i^*)) is discrete.

If we now write each \( A_i^* \) and \((U_i, A_i^*)'\) as the union of convex components, \( A_i^* = U_a A_i^a \) and \((U_i, A_i^*)' = U_y C_y \), the collection \( M = \{A_i^a, C_y\} \) inherits a linear order from \( X \) and is thus itself a linearly ordered set. We claim that in the ordered set \( M \), each of the sets \( A_i^a \) has an immediate successor whenever \( A_i^a \) intersects the closure of \( S_i^a \), the set of strict upper bounds for \( A_i^a \). For suppose \( A_i^a \cap \text{cl}(S_i^a) \neq \emptyset \). Then \( A_i^a \cap \text{cl}(S_i^a) \)

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contains precisely one point, say \( p \), every neighborhood of which intersects \( A_i \). Thus since \( \text{cl}(A_i) \) is discrete, there exists a neighborhood \( (x, y) \) of \( p \) disjoint from \( \bigcup_{i \neq j} \text{cl}(A_i) \). Then \( (x, y) \cap S_a \neq \emptyset \), so \( (p, y) \neq \emptyset \). But the definition of \( A_i \) insures that \( (p, y) \) is disjoint from both \( A_i \) and \( \bigcup_{i \neq j} A_j \), so there must exist some set \( C_y \) containing \( (p, y) \). In the linear order on \( M \), \( C_y \) is the immediate successor to \( A_i \), and we will call it \( C_{i_y} \).

For each \( y \) select and fix some point \( k_{\gamma} \in C_y \). Then whenever \( A_i \cap \text{cl}(S_{i} \neq \emptyset \), there exists a unique \( k_{i_y} \in C_{i_y} \), the immediate successor of \( A_i \). In such cases, let \( I_{i} = [p, k_{i_y}] \) where \( p \in A_i \cap \text{cl}(S_{i} \); otherwise, if \( A_i \cap \text{cl}(S_{i} = \emptyset \), let \( I_{i} = \emptyset \). Define \( J_{i} \) similarly for the strict lower bounds of \( A_i \) (using the same collection of points \( k_{i} \in C_{i} \)). Then for each \( a \) and each \( i \), let \( U_{i} = J_{i} \cup A_i \cup I_{i} \). Each \( U_{i} \) is clearly an open set containing \( A_i \), so \( U_{i} = U_{i} \cup U_{i} \) is an open set containing \( A_i \). Since no \( A_i \) intersects any \( A_j \) for \( i \neq j \), and since the use of the same \( k_{i} \) throughout implies that no \( J_{i} \) or \( I_{i} \) may intersect any \( J_{j} \) or \( I_{j} \), it is clear that no \( U_{i} \) can intersect any \( U_{j} \) for \( i \neq j \). Thus \( U_{i} \cap U_{j} = \emptyset \) whenever \( i \neq j \), and hence \( X \) is collectionwise normal.

Now every subspace of \( X \) inherits both a topology as well as a linear order; these need not be compatible, even for open subspaces. (The open subspace \( \{\alpha + 1 \mid \alpha \text{ is a limit ordinal}\} \) of the linearly ordered ordinal space \( \{\gamma \mid \gamma < \Omega\} \) inherits the discrete topology but is of the same order type as the countable ordinals.) However, the two structures are compatible on convex subspaces of \( X \), whence convex subspaces of \( X \) are collectionwise normal. Therefore any open subset of \( X \)—being the disjoint union of open collectionwise normal subspaces (namely its convex components)—is collectionwise normal. This suffices to prove that every subset \( S \) of \( X \) is collectionwise normal, since if \( \{A_i\} \) is a discrete family in \( S \), then each point \( s \in S \) has a neighborhood \( U_s \cap S \) which meets at most one of the sets \( A_i \). But then \( U = U_s \cup U_s \) is an open set with the same property, and since \( U \) is collectionwise normal, so must be \( S \). Hence \( X \) is hereditarily collectionwise normal.

That \( X \) is completely normal (i.e., hereditarily normal) follows as a corollary. But it also may be proved more directly by a slight modification of the proof that \( X \) is collectionwise normal.

**References**


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