OSCILLATION CRITERIA FOR NONLINEAR MATRIX DIFFERENTIAL INEQUALITIES

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Abstract. Oscillation criteria are established for nonlinear matrix differential equations of the form 
\[ [A(x)V']' + B(x, V, V')V = 0 \]
and associated differential inequalities. The hypothesis used recently by E. C. Tomastik, that \( A \) and \( B \) are positive definite, is weakened to the following: \( A \) is positive semidefinite.

Oscillation criteria for the matrix differential equation

\[
LV = [A(x)V']' + B(x, V, V')V = 0,
\]
and more generally for the inequality \( V^T LV \leq 0 \) (as a form), will be derived by a technique different from that given recently by Tomastik \[3\]. It will be assumed that \( A, B, \) and \( V \) are \( m \times m \) matrix functions, \( A(x) \) is symmetric, positive semidefinite, and continuous on an interval \([a, \infty)\), and \( B(x, V, V') \) is symmetric and continuous for \( x \) on \([a, \infty)\) and for all values of the entries of \( V \) and \( V' \). Although Tomastik requires \( B(x, V, V') \) to be positive definite on \( a \leq x < \infty \) for every matrix \( V \) with \( \det V \neq 0 \), we require only that \( B(x, V, V') \) satisfies condition (2) below. As already noted, we have weakened the positive definiteness of \( A(x) \) to positive semidefiniteness. The technique used here has the advantage that it can be adapted \[2\], \[1\], Chapter 5] to partial differential inequalities of elliptic or parabolic type. As in \[3\] it is assumed that every solution of (1) can be continued to \( x = \infty \).

Theorem 1. The inequality \( V^T LV \leq 0 \) is oscillatory if \( A(x) \) is bounded above and there exists a diagonal element \( B_{ii} \) of \( B \) such that

\[
\int_a^\infty B_{ii}[x, V(x), V'(x)]dx = +\infty
\]

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for every differentiable matrix \( V(x) \) with \( \det V(x) \neq 0 \) for all sufficiently large \( x \).

**Proof.** Suppose to the contrary that \( V^T L V \leq 0 \) is not oscillatory, i.e. [3] there exists a number \( b \geq a \) and a prepared matrix \( V(x) \) satisfying \( V^T L V \leq 0 \) such that \( \det V(x) \neq 0 \) in \( (b, \infty) \). Then a unique solution \( w(x) \) of \( u(x) = V(x)w(x) \) exists in \( (b, \infty) \) for any \( m \)-vector \( u(x) \). The following identity is easily verified by differentiation for any piecewise \( C^1 \) vector function \( u \):

\[
(Vw)^T A V w' + [(Vw)^T A V' w]' = u^T A u' - u^T B u + u^T L V w + w^T (V^T A V' - V^T A V) w'.
\]

The last term is identically zero since \( V \) is prepared [3]. Since \( V^T L V \leq 0 \), it follows that

\[
F[u, V] = \int_b^c [u^T A(x) u' - u^T B(x, V, V')] dx \geq 0
\]

if \( u \) is any piecewise \( C^1 \) vector function satisfying \( u(b) = u(c) = 0 \) \( (b < c) \). In particular, choose \( u \) to be \( u_i \), where

\[
\begin{align*}
u_{i}(x) = 0 & \quad \text{if } a \leq x \leq b \\
= e_i(x - b) & \quad \text{if } b < x \leq b + 1 \\
= e_i & \quad \text{if } b + 1 < x \leq c - 1 \\
= e_i(c - x) & \quad \text{if } c - 1 < x \leq c
\end{align*}
\]

\( (c > b + 2) \) and \( e_i \) is the unit vector with 1 in the \( i \)th position and 0 elsewhere. Then

\[
F(u_i, V) \leq 2\alpha - \beta^3 - \int_{b+1}^{c-1} B_{ii}(x, V, V') dx
\]

\[
- \int_{c-1}^c B_{ii}(x, V, V')(c - x)^2 dx
\]

where \( \alpha \) is an upper bound for \( A(x) \) on \( [b, \infty) \) and \( \beta \) is a lower bound for \( B(x, V(x), V'(x)) \) on \( [b, b+1] \). In view of the hypothesis (2), there exists a number \( c_0 \) such that

\[
F(u_i, V) \leq - \int_{c-1}^c B_{ii}(x, V, V')(c - x)^2 dx
\]

for all \( c \geq c_0 \). Define
We assert there exists a number $c \geq c_0$ such that $g(c - 1, x) > 0$ for all $x > c - 1$. In fact, $g(c_0 - 1, x)$ has a largest zero $x = c - 1$ by (2) ($c \geq c_0$) and $g(c_0 - 1, x) = g(c - 1, x) > 0$ for $x > c - 1$. An easy integration by parts shows that

$$\int_{c-1}^{c} B_{ii}(x, V, V')(c - x)^2 dx = 2 \int_{c-1}^{c} (c - x)g(c - 1, x)dx > 0.$$ 

Hence $F[u, V] < 0$, contradicting (3).

Theorem 1 remains true if (2) is replaced by the obviously stronger condition

$$\int_{a}^{\infty} \text{tr} \, B[x, V(x), V'(x)] dx = + \infty,$$

where $\text{tr} \, B$ denotes the trace of $B$. In the case that $B$ is positive definite, Theorem 1 also remains true if (2) is replaced by the equivalent condition

$$\int_{a}^{\infty} \lambda[x, V(x), V'(x)] dx = + \infty$$

where $\lambda$ denotes the largest eigenvalue of $B$. In general, it is clear that (4) can be replaced by (5) provided also $\int_{a}^{\infty} \lambda_i(x)dx$ is finite for every negative eigenvalue $\lambda_i$ of $B[x, V(x), V'(x)]$.

If $A$ is the identity matrix and $B$ is positive definite, the hypothesis (5) is needed only for matrices $V(x)$ such that the smallest eigenvalue of $V^T(x)V(x)$ is bounded away from zero for large $x$, as shown by Tomastik [3].

**Corollary.** The equation $LV = 0$ is oscillatory if $A$ and $B$ are positive definite, $A$ is bounded above, and (5) holds for every differentiable matrix $V$ with $\det V(x) \neq 0$ for all sufficiently large $x$.

This is an obvious special case of Theorem 1; it is also a specialization of Tomastik's Theorem 3 to the case that $A$ is bounded above.

The following generalization of Theorem 1 can be proved in the same way.

**Theorem 2.** $V^T LV \leq 0$ is oscillatory if for arbitrary $b \geq a$ there exists a number $c \ (c > b)$, an integer $i$, and a piecewise $C^1$ function $\phi$ on $[b, c]$ such that $\phi(b) = \phi(c) = 0$ and
\[
\int_b^c \left\{ \phi''(x) A_{ii}(x) - \phi''(x) B_{ii}[x, V(x), V'(x)] \right\} dx < 0
\]

for every differentiable matrix \( V(x) \) with \( \det V(x) \neq 0 \) on \([b, \infty)\).

**References**


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