Let $E$ be a subset of the complex plane. For $m$ positive define

$$A(E, m) = \{ f : f \text{ is analytic off a compact subset of } E, f(\infty) = 0, ||f|| \leq m \}.$$ 

For $f \in A(E, m)$, the derivative at $\infty$ of $f$ is its first Laurent coefficient $f'(\infty) = \lim_{z \to \infty} zf(z)$. The analytic capacity of $E$ is defined as

$$\gamma(E) = \sup \{ |f'(\infty)| : f \in A(E, 1) \}.$$

Of the several uses of analytic capacity we only mention one: that $\gamma(E) = 0$ if and only if every bounded analytic function on the complement of $E$ is constant.

The length, or one-dimensional Hausdorff measure, of a set $E$ is $l(E) = \lim_{\rho \to 0} \Lambda_\rho(E)$, where

$$\Lambda_\rho(E) = \inf \{ \sum \delta_j : E \subseteq \bigcup \Delta(a_j, \delta_j), \delta_j \leq \rho \}$$

and $\Delta(a_j, \delta_j)$ is the disc $\{ |z - a_j| < \delta_j \}$. When $E$ lies on a rectifiable curve, this notion is equivalent to that of the (outer) arc length of $E$.

A classical theorem of Painlevé is that $\gamma(E) = 0$ whenever $l(E) = 0$. When $E$ lies on a sufficiently smooth curve, $\gamma(E)$ and $l(E)$ can only vanish simultaneously [2]. On the other hand, A. G. Vituškin [3] has given an example of a set $E$ with $l(E) > 0$ but $\gamma(E) = 0$. However Vituškin’s proof is quite complicated and contains many typographical errors. We give a simpler counterexample, and compare ours to Vituškin’s.

1. The example. The example is the planar Cantor set obtained by taking the “corner quarters.” Let $K = \cap_{n=0}^\infty E_n$ where $E_0$ is the unit square, $E_n$ consists of $4^n$ squares of side $4^{-n}$, and each component of $E_n$ contains four components of $E_{n+1}$, these being the four corner squares of side $4^{-n-1}$. The components of $E_n$ will be indexed as $E_{n,j}$, $1 \leq j \leq 4^n$. Set $K_{n,j} = K \cap E_{n,j}$.

It is obvious that $\Lambda(K) = 2^{-1/2}$.

Our proof that $\gamma(K) = 0$ resembles Vitushkin’s argument but is simpler because it takes advantage of the homogeneity of the set $K$. That is, $K_{n,j}$ is geometrically similar to $K$, so that one can make linear changes of variable, and so that $\gamma(K_{n,j}) = 4^{-n}\gamma(K)$. 

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Let $f \in A(K, 1)$, and assuming $\gamma(K) > 0$, suppose $a = f'(\infty)$ is real and positive. For $z \in K$, let $\Gamma_{n,j}$ be a cycle with winding number one about $K_{n,j}$ but zero about $K - K_{n,j}$ and about $z$, and set

$$f_{n,j}(z) = \frac{-1}{2\pi i} \int_{\Gamma_{n,j}} \frac{f(w)dw}{w - z}.$$

**Lemma 1.**

(a) $\sum_{j=1}^{4^n} f_{n,j} = f$.

(b) There is a constant $M$ such that $f_{n,j} \in A(K_{n,j}, M)$.

(c) $|f_{n,j}(\infty)| \leq M 4^{-n} \gamma(K)$.

**Proof.** The Cauchy integral theorem yields (a) as well as the fact that $f_{n,j}$ is analytic off $K_{n,j}$. If $V$ is a square concentric with $E_{n,j}$ but three times as large, then for $z$ near $K_{n,j}$ we have

$$f(z) - f_{n,j}(z) = \frac{1}{2\pi i} \int_{\partial V} \frac{f(w)dw}{w - z},$$

so that $|f_{n,j}(x)| \leq 1 + 6/\pi$. (c) follows directly from (b).

Set $a_{n,j} = f_{n,j}(\infty)$.

**Lemma 2.** Let

$$h_{n,j}(z) = a_{n,j} A_{2n} \int_{\partial E_{n,j}} \frac{dxdy}{x + iy - z}$$

for $z \in E_{n,j}$, and set $h_n = \sum_{j=1}^{4^n} h_{n,j}$. Then the $h_n$ are uniformly bounded.

**Proof.** A well-known estimate using polar coordinates and part (c) of Lemma 1 yield the estimate $|h_{n,j}(z)| \leq M \gamma(K) A_{2n} = M_1$. Also $h_{n,j}(\infty) = -a_{n,j}$. If $g_{n,j} = f_{n,j} + h_{n,j}$; then $|g_{n,j}| \leq M + M_1 = M_2$ and $g_{n,j}$ vanishes twice at $\infty$. Two applications of Schwarz's lemma then give

$$|g_{n,j}(z)| \leq \frac{M_2 A_{2n}^2}{(\text{dist}(z, E_{n,j}))^2}.$$

As $f = \sum f_{n,j}$, we must show $\sum_{j=1}^{4^n} g_{n,j}$ is uniformly bounded. Now for $z$ close to $E_{n,j_0}$ we have

$$\sup_{z \in E_{n,j}} \sum_{j} |g_{n,j}(z)| \leq \sup_{z \in E_{n,j}} \left( 1 + \sum_{j \neq j_0} \frac{M_2 A_{2n}^2}{(\text{dist}(z, E_{n,j}))^2} \right) = B_n.$$

To estimate $B_n$, compare the $4^{n-1}$ terms corresponding to the $4^{n-1}$ squares in the same component of $E_1$ as $z$ with the supremum $B_{n-1}$, and estimate the remaining terms by $4 M_2 A_{2n-1}$. This gives $B_n \leq B_{n-1} + 3 \cdot 4^{n-1} \cdot 4 M_2 A_{2n-1}^2$, so that $\lim B_n < \infty$, and the $h_n$ are uniformly bounded.
Actually one can show that \( f = -\lim_n h_n \), but we will not need this. We finish the proof by showing that the case \( a_{n,j} = a 4^{-n} \) for all \( j \) is impossible, but that otherwise Lemma 1(c) is contradicted.

**Lemma 3.** For some \( n \) and \( j \), \( a_{n,j} \neq a 4^{-n} \).

**Proof.** Assume \( a_{n,j} = a 4^{-n} \) for all \( n \) and \( j \). Then \( A_n = a^{-1} \text{Re}(h_n(0)) \) is bounded. Now

\[
A_n = 4^n \sum_{j=1}^n \int \int_{B_{n,j}} \frac{x \, dx \, dy}{x^2 + y^2}.
\]

But

\[
4^n \sum_{k \in \{a \neq 1\}} \int \int_{B_{n,j}} \frac{x \, dx \, dy}{x^2 + y^2} \geq 4^{-2}
\]

while the sum of the \( 4^{n-1} \) lower left integrals is \( A_{n-1} \). Thus \( A_n \geq A_{n-1} + 4^{-2} \), a contradiction.

**Lemma 4.** For any \( \varepsilon > 0 \) and any \( M > 0 \), there exists \( \delta > 0 \) such that for any \( f \in A(K, M) \) with \( |f'(\infty)| \geq \varepsilon \), we have

\[
\sup_{n,j} 4^n |a_{n,j}| \geq (1 + \delta) |f'(\infty)|.
\]

**Proof.** Suppose not. Then for \( \delta_k \downarrow 0 \) there exists \( f_k \in A(K, M) \) with \( |f'_k(\infty)| \geq \varepsilon \) such that \( \|a_{n,j}^{(k)}\| \leq 4^{-n} |f'_k(\infty)| \). This means \( a_{n,j} = a(1 + \delta_k) f'_k(\infty) \), which is impossible by Lemma 3.

Finally, to show \( \gamma(K) = 0 \), let \( f \in A(K, 1) \) with \( a = f'(\infty) > 0 \). Choose \( n_1 \) and \( j_1 \) such that by Lemma 4 (with \( \varepsilon = a \) and \( M \) as in Lemma 1),

\[
|a_{n_1,j_1}| \geq a(1 + \delta) 4^{-n_1}.
\]

Since \( K_{n_1,j_1} \) is geometrically similar to \( K \) we can apply Lemma 4 to \( f_{n_1,j_1} \). Continuing, we obtain a sequence \( (n_k, j_k) \) with \( |a_{n_k,j_k}| \geq a(1 + \delta)^k 4^{-n_k} \). This contradicts Lemma 1(c).

2. **Comparison with Vitushkin’s example.** Take a nondecreasing sequence of positive integers \( n_j \) with \( n_1 \geq 2 \). Set \( E_0 = [0, 1], E_1 = \bigcup_{k=1}^{n_1} \{1/k\} \times [0, 1/n_1] \). Obtain \( E_{k+1} \) from \( E_k \) by repeating this process with each interval in \( E_k \) but using \( n_{k+1} \). Set \( E = \lim_k E_k \). For any choice of \( n_j \), \( l(E) \) is positive. Vitushkin [3] shows that \( \gamma(E) = 0 \).
if \( \lim_{j \to \infty} n_j = \infty \). Setting \( r_k = \prod_{j=1}^{k} n_j^{-1} \), we see that \( E_k \) consists of pairwise disjoint intervals of length \( r_k \). Replacing each interval by a rectangle of sides \( r_k \) and \( r_{k+1} \), we have \( E = \bigcap E_k \). Then the reasoning of §1 above shows that \( \gamma(E) = 0 \) if \( n_j = m \) for all sufficiently large indices \( j \).

3. Remarks. It is interesting to compare our example with the known results for similar Cantor sets. Let \( C_r \) \( (0 < r < 1) \) be the Cantor set on \([0, 1]\) obtained by removing \( r \)-ths, and let \( K_r = C_r \times C_r \). Thus our set is \( K_{1/2} \). For \( r > 1/2 \), \( l(K_r) = 0 \), and thus \( \gamma(K_r) = 0 \). For \( r < 1/2 \), Denjoy [1] proved that \( \gamma(K_r) > 0 \). Indeed, Denjoy constructed a function in \( A(K, 1) \) which extended continuously to the entire plane. In other words, if \( r < 1/2 \), \( K_r \) has positive continuous analytic capacity: \( \alpha(K_r) > 0 \). It was also known [4], that \( \alpha(K_{1/2}) = 0 \), because \( l(K_{1/2}) < \infty \). Thus our example completes the study of analytic capacity for such Cantor sets.

REFERENCES


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