ON THE FACTORIZATION OF INTEGERS

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ABSTRACT. The order of magnitude of the average of the exponents in the canonical factorization of an integer is discussed. In particular, it is shown that this average has normal order one and a result which implies that the average order is one is also derived.

Let $n = p_1^{a_1} \cdots p_r^{a_r}$ be the factorization of n as a product of powers of distinct primes. Our purpose is to consider the average of the exponents in this decomposition. Let $\omega(n) = r$; $\Omega(n) = a_1 + \cdots + a_r$; and $a(n) = \Omega(n)/\omega(n)$. Since $\omega(n)$ and $\Omega(n)$ both have normal order $\log\log n$ [1], it is obvious that $a(n)$ has normal order one and it will be shown that $a(n)$ has average order one. Similar results for the minimum and maximum exponents in the above factorization have been given recently by Niven [2].

LEMMA 1. $\sum_{n \leqq x} 1/\omega(n) = O(x/\log\log x)$.

PROOF.

$$\begin{aligned}\sum_{n \leqq x} 1/\omega(n) &= \sum_{n \leqq x;\, 2\omega(n) < \log\log x} 1/\omega(n) + \sum_{n \leqq x;\, 2\omega(n) \geqq \log\log x} 1/\omega(n) \\ &\leqq \sum_{n \leqq x;\, 2\omega(n) < \log\log x} 1 + 2x/\log\log x \\ &\leqq (4C+2)x/\log\log x,\end{aligned}$$

since by Turan's inequality [1]

$$\begin{aligned}Cx \log\log x &\geqq \sum_{n \leqq x;\, 2\omega(n) < \log\log x} [\omega(n) - \log\log x]^2 \\ &\geqq \tfrac{1}{4}(\log\log x)^2 \sum_{n \leqq x;\, 2\omega(n) < \log\log x} 1.\end{aligned}$$

LEMMA 2.

$$\sum_{n \leqq x} a(n) = x + O\left[\sum_{n \leqq x} 1/\omega(n) + x^{1/2} \int_1^x t^{-3/2} \sum_{n \leqq t} 1/\omega(n)\, dt\right].$$

PROOF.

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$$\begin{aligned}\sum_{n\leq x} a(n) &= \sum_{n\leq x} \Omega(n)/\omega(n) = \sum_{n\leq x}\sum_{p^m|n} 1/\omega(n)\\ &= \sum_{n\leq x}\sum_{p|n} 1/\omega(n) + \sum_{n\leq x}\sum\nolimits^*_{p^m|n} 1/\omega(n)\\ &= \sum_{n\leq x} 1 + \sum\nolimits^*_{p^m\leq x}\sum_{n\leq xp^{-m}} 1/\omega(np^m)\\ &= x + O\Big[\sum\nolimits^*_{p^m\leq x}\sum_{n\leq xp^{-m}} 1/\omega(n)\Big]\\ &= x + O\Big[\sum_{n\leq x}\sum\nolimits^*_{p^m\leq x/n} 1/\omega(n)\Big]\end{aligned}$$

where $\sum^*$ denotes a sum over $m\geq 2$. Now $\sum^*_{p^m\leq t} 1 = O(t^{1/2})$ by Tchebycheff's inequality and the desired result follows by partial summation.

The following theorem, which implies that $a(n)$ has average order one, is an immediate consequence of the above lemmas.

THEOREM. $\sum_{n\leq x} a(n) = x + O(x/\log\log x)$.

REFERENCES

1. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Oxford Univ. Press, London, 1960.

2. Ivan Niven, *Averages of exponents in factoring integers*, Proc. Amer. Math. Soc. 22 (1969), 356–360.

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