ON THE ASYMPTOTIC BEHAVIOR OF LINEAR SYSTEMS

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ABSTRACT. The purpose of this paper is to establish a necessary and sufficient condition for the vector-matrix system $\dot{x} = [A(t) + B(t)]x$ to have solutions of the form $Y(t)c(t)$ where $Y(t)$ is a fundamental matrix of solutions of $\dot{y} = A(t)y$.

Consider the linear systems

$$\dot{y} = A(t)y, \tag{1}$$

$$\dot{x} = [A(t) + B(t)]x, \tag{2}$$

where x, y are n-vectors and $A(t)$, $B(t)$ are continuous $n \times n$ matrices on $[t_0, \infty)$. Let $Y(t)$ be a fundamental matrix of solutions of (1) and put $Z(t) = Y^{-1}(t)B(t)Y(t)$. Let $\|\ \|$ denote any appropriate vector-matrix norm. We establish the following result.

THEOREM. *Given an arbitrary constant n-vector c, equation* (2) *has a unique solution x(t) of the form*

$$x(t) = Y(t)c(t), \tag{3}$$

where c(t) is an n-vector function satisfying

$$\lim_{t \to \infty} c(t) = c, \qquad \int_{t_0}^{\infty} \|\dot{c}(t)\| dt < \infty \tag{4}$$

if and only if

$$\int_{t_0}^{\infty} \|Z(t)\| dt < \infty. \tag{5}$$

REMARK. The sufficiency of condition (5) has been established by Bebernes and Vinh [1]. (Bebernes and Vinh do not mention that $\int_{t_0}^{\infty} \|\dot{c}(t)\| dt < \infty$, but it follows readily from their proof.) Our proof of the sufficiency of condition (5), while similar to theirs, differs in several significant details and is therefore included below.

PROOF. First assume that (5) holds. Let c be a given n-vector. Let $X(t)$ be a fundamental matrix solution of (2) and let the matrix $C(t)$ be defined by the "variation of parameter" equation $X(t) = Y(t)C(t)$. It is easily shown that $C(t)$ satisfies

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(6) $$\dot{C}(t) = Z(t)C(t).$$

Integrating and applying Gronwall's inequality, it follows that $\|C(t)\|$ is bounded on $[t_0, \infty)$. Hence (5) and (6) imply that $\int_{t_0}^{\infty}\|\dot{C}(t)\|dt<\infty$. Hence $\int_{t_0}^{\infty}\dot{C}(t)dt$ converges, whence $\lim_{t\to\infty} C(t)$ exists. Denote this limit by C. By Liouville's theorem

$$\det C(t) = \det C(t_0) \exp \int_{t_0}^{t} \text{trace } Z(s)ds.$$

Certainly $\det C(t_0)\neq 0$; by (5) we can let $t\to\infty$, concluding that $\det C \neq 0$. Take $c(t)=C(t)C^{-1}c$, $x(t)=X(t)C^{-1}c$. The uniqueness of $c(t)$ is readily established.

Conversely, suppose that given any n-vector c, (2) has a (unique) solution $x(t)$ of the form (3) satisfying (4). Thus, in particular, given a nonsingular square matrix C, (2) has a (fundamental) matrix solution $X(t)$ of the form $X(t)=Y(t)C(t)$ where $C(t)$ satisfies $\lim_{t\to\infty} C(t)=C$, $\int_{t_0}^{\infty}\|\dot{C}(t)\|dt<\infty$. Since $C(t)$ must satisfy (6) and since $\lim_{t\to\infty}\det C(t) =\det C\neq 0$, it follows that $\det C(t)\neq 0$ on $[t_0, \infty)$. Thus $\lim_{t\to\infty} C^{-1}(t) = C^{-1}$, whence $\|C^{-1}(t)\|$ is bounded on $[t_0, \infty)$. Since $Z(t)=\dot{C}(t)C^{-1}(t)$, it follows that condition (5) holds. This completes the proof.

Related results for scalar equations may be obtained as corollaries. In particular, consider the nth-order linear scalar equations

(1*) $$v^{(n)} = \sum_{j=0}^{n-1} a_j(t)v^{(j)},$$

(2*) $$u^{(n)} = \sum_{j=0}^{n-1} [a_j(t) + b_j(t)]u^{(j)},$$

where the coefficient functions are continuous on $[t_0, \infty)$. Let $v_1(t), \cdots, v_n(t)$ be a fundamental set of solutions of (1*). Let $W(t)$ denote their Wronskian. Let $W_k(t)$ $(k=1, \cdots, n)$ be the determinant obtained by replacing all elements in the kth column of $W(t)$ by zero except the element in the nth row which is replaced by 1. Let

$$M(t) = \max_{1\leqq k\leqq n}\left|\sum_{j=0}^{n-1} b_j(t)v_k^{(j)}(t)\right|.$$

We have the following result.

COROLLARY. *Assume*

(7) $$\int_{t_0}^{\infty} | a_{n-1}(t) | \, dt < \infty.$$

Given arbitrary constants $c_1, \cdots, c_n$, *equation* (2*) *has a unique solution* $u(t)$ *of the form*

$$u^{(j)}(t) = \sum_{k=1}^{n} c_k(t) v_k^{(j)}(t), \qquad j = 0, \cdots, n-1, \tag{3*}$$

satisfying

$$\lim_{t\to\infty} c_k(t) = c_k, \qquad \int_{t_0}^{\infty} |\dot{c}_k(t)|\, dt < \infty \tag{4*}$$

if and only if

$$\int_{t_0}^{\infty} |W_k(t)|\, M(t)dt < \infty, \qquad k = 1, \cdots, n. \tag{5*}$$

PROOF. Equations (1*) and (2*) are easily seen to be equivalent, respectively, to linear systems of the form (1) and (2) with

$$A(t) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ a_0(t) & a_1(t) & \cdots & a_{n-1}(t) \end{bmatrix},$$

$$B(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ b_0(t) & b_1(t) & \cdots & b_{n-1}(t) \end{bmatrix}.$$

Condition (5) is found to be equivalent to $\int_{t_0}^{\infty} (|W_k(t)|/|W(t)|)M(t)dt < \infty$, $k=1, \cdots, n$. Liouville's theorem, together with (7), implies that the above condition is equivalent to (5*).

REMARK. The above corollary sharpens a result of Katz [**2**] who assumes that $a_{n-1}(t) \equiv b_{n-1}(t) \equiv 0$ on $[t_0, \infty)$, a stronger requirement than (7).

REFERENCES

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