AN UNSOLVABLE EQUATION[1]

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ABSTRACT. A linear partial differential equation that is not solvable is obtained by a very simple transformation of the Cauchy-Riemann equations.

It seems worth while to call attention to the following simple example, which is implicit in the work of Lewy [1] and Nirenberg and Treves [2]:

THEOREM. *The linear partial differential equation*

$$u_x + ixu_y = |xy|$$

is not classically solvable in any neighborhood of the origin.

PROOF. Since the odd part with respect to x of any solution is also a solution, the expression

$$U(x, y) = u(x, y) - u(-x, y) + iy|y|$$

satisfies the homogeneous equation

$$U_x + ixU_y = 0$$

for $x > 0$. Therefore it is an analytic function of the complex variable $z = x^2/2 + iy$ in some neighborhood of the origin within the right half-plane, and moreover its real part vanishes on the imaginary axis. By the Schwarz principle of reflection U can be continued analytically into the left half-plane, so $U(0, y)$ must be an analytic function of y for small $|y|$. But this contradicts the fact that the second derivative of $y|y|$ is discontinuous at the origin.

Smoother inhomogeneous terms such as $|xy|^3$ or $\exp(-1/y^2)$ provide unsolvable equations that are almost as easy to analyze. More specifically, in the harder case of $\exp(-1/y^2)$ one has only to consider the analytic function

$$U\left(\frac{x^2}{2} + iy\right) = \frac{1}{\pi i} \int_{-x}^{x} \frac{u(\xi, y)\xi d\xi}{(x^2 - \xi^2)^{1/2}} + \int \exp(-1/y^2)dy.$$

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Also, without looking at such a complicated integral it is easy to see that the infinitely differentiable equation

$$u_x + i(xe^{-1/x^2}/|x|)u_y = \exp(-1/x^2 - 1/y^2)$$

is not solvable at the origin.

REFERENCES

1. H. Lewy, *An example of a smooth linear partial differential equation without solution*, Ann. of Math. (2) **66** (1957), 155–158. MR **19**, 551.

2. L. Nirenberg and F. Treves, *Solvability of a first order linear partial differential equation*, Comm. Pure Appl. Math, **16** (1963), 331–351. MR **29** #348.

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