THE ALGEBRA OF LOG-SUMMABLE FUNCTIONS

DANIEL O. ETTER, JR.

ABSTRACT. The space L_0 consists of measurable functions f on $[0, 1]$ such that $\log(1+|f(x)|)$ is summable on $[0, 1]$, with functions equal almost everywhere identified. The integral defines a quasinorm on L_0. With this quasinorm, L_0 becomes a complete quasinormed linear space, the topology of which is not locally bounded. The quasinorm is plurisubharmonic (subharmonic on one-dimensional complex manifolds). L_0 is closed under multiplication, and multiplication is continuous. Inversion is not continuous, and the group of invertible elements is not open. There are no proper closed maximal ideals. The resolvent $(\lambda - f)^{-1}$ may exist for all complex λ, but it cannot be entire.

Introduction. The space L_0 of log-summable functions is defined by Dunford and Schwartz [1, pp. 534–535]. For the particular case of the unit interval $[0, 1]$ with Lebesgue measure, the definition is as follows. A measurable function f is said to belong to L_0 if and only if

$$-\infty \leqq \int_0^1 \log|f(x)|\,dx < \infty.$$

No topology is given in [1] for L_0. A topology can be defined by means of the functional q_0:

$$q_0(f) = \int_0^1 \log(1 + |f(x)|)dx.$$

The functional q_0 is a quasinorm for L_0 which is plurisubharmonic (i.e., subharmonic on one-dimensional complex manifolds).

There are two reasons for calling attention to (L_0, q_0). First, it is an example of a linear space with a plurisubharmonic quasinorm, the topology of which is not locally bounded. Previously known linear spaces with plurisubharmonic quasinorms included only Banach spaces and L_p and H_p spaces for $0<p<1$ [2]; these all have locally bounded topologies. Second, (L_0, q_0) is a topological algebra which fails to have any of the properties usually associated with spectral

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theoretic results. Many elements of L_0 do have empty spectra. But, even though the resolvent $r(\lambda, f) = (\lambda - f)^{-1}$ may belong to L_0 for all λ in the complex plane, the set on which $r(\lambda, f)$ is analytic cannot be the entire complex plane. For an essentially bounded measurable function f, with $q_\infty(f) = \text{ess sup } \{ |f(x)| : 0 \leqq x \leqq 1 \}$, the resolvent $r(\lambda, f)$ is analytic for $|\lambda| > q_\infty(f)$.

Basic properties of (L_0, q_0). In this section, some basic properties of (L_0, q_0) as a quasinormed linear space are enumerated.

Convention. Functions equal almost everywhere are identified.

PROPOSITION 1. *Let a denote a complex scalar, f and g denote functions in L_0, and $\{f_m : m = 1, 2, 3, \cdots\}$ a sequence of functions in L_0. Then the functional q_0 has the following properties:*

(i) *If $q_0(f_m) \to 0$ then $f_m \to 0$ in measure.*

(ii) *q_0 is subadditive and the function $(a, f) \to q_0(af)$ is hypocontinuous.*

(iii) $q_0(af) = q_0(|a| f)$.

(iv) $q_0(fg) \leqq q_0(f) + q_0(g)$.

PROOF. Only (ii) may require comment. Subadditivity of q_0 is obvious, e.g., from $\log(1+u) = \int_0^u (1+x)^{-1} dx$, for $u > 0$. The dominated convergence theorem shows that $q_0(af) \to 0$ if $a \to 0$; the same theorem, together with subadditivity of q_0, shows that if $q_0(f_m) \to 0$ then $q_0(af_m) \to 0$, so that hypocontinuity (continuity in each variable separately) follows.

A standard proof that L_p is complete [4, p. 231, p. 243] can be modified easily to show that (L_0, q_0) is complete. It then follows that $q_0(af)$ is continuous in a and f simultaneously, and so q_0 is a quasinorm (q_0 is subadditive, and $(a, f) \to q_0(af)$ is continuous).

PROPOSITION 2. *The space (L_0, q_0) is a complete quasinormed linear space.*

PROPOSITION 3. *The injection mapping $J : L_p \subset L_0$ is continuous for $0 < p \leqq \infty$.*

PROOF. Dunford and Schwartz state that $L_p \subset L_0$ [1, p. 535, Problem 33]. Continuity of J for $0 < p \leqq 1$ is obvious, since $p \log(1+u) = \log(1+u)^p \leqq \log(1+u^p) \leqq u^p$ for $u \geqq 0$ and $0 < p \leqq 1$. The extension to all $p > 0$ is then immediate.

COROLLARY. *The space L_0 has no nonzero continuous linear functionals.*

COROLLARY. *The topology of L_0 is not locally convex.*

PROPOSITION 4. *The topology of L_0 is not locally bounded.*

PROOF. It suffices to verify that for every $\epsilon > 0$ the sphere of radius ϵ around 0 in L_0 contains a topologically unbounded sequence. Consider the functions:

$$f_{m,a}(x) = ax^{-m}, \quad \text{for } a > 0 \text{ real, } m > 0 \text{ an integer.}$$

One has

$$q_0(f_{m,a}) = \int_0^1 \log(1 + ax^{-m})dx$$
$$= \log(1 + a) + m \int_0^1 ax^{-m}/(1 + ax^{-m})dx.$$

By substituting $y = ax^{-m}$, one obtains

$$q_0(f_{m,a}) = \log(1 + a) + a^{1/m} \int_a^\infty [y^{1/m}(1 + y)]^{-1}dy.$$

It is clear that:

$$\int_0^\infty [y^{1/m}(1 + y)]^{-1}dy \to \infty \quad \text{as } m \to \infty.$$

Thus, given $\epsilon > 0$ there exist sequences $m_j \to \infty$ and $a_j \to 0$ such that, if $h_j = f_{m_j, a_j}$, then $q_0(h_j) < \epsilon$ for all j and $\lim_{j\to\infty} q_0(h_j) = \epsilon/2$, say. It is also easily found that:

$$\lim_{j\to\infty} q_0(\lambda h_j) = \lim_{j\to\infty} \lambda^{1/m_j} q_0(h_j) = \epsilon/2.$$

The sequence $\{h_j : j = 1, 2, 3, \cdots\}$ is therefore not topologically bounded, since it cannot be absorbed by a sphere of radius less than $\epsilon/2$.

The properties of L_0 thus resemble closely those of the space M of measurable functions, given the topology of convergence in measure. The results of the next section heighten the similarity.

Algebraic properties. It is clear that L_0 is a commutative linear algebra. It may be a little unexpected that multiplication is continuous.

LEMMA. *Multiplication in L_0 is continuous in each variable separately.*

PROOF. This follows from the closed graph theorem. Specifically, let $f \in L_0$, $f_m \in L_0$, and $g \in L_0$, and suppose $f_m \to f$ and $f_m g \to h \in L_0$. Then $f_m g \to h$ in measure, so $h = fg$ almost everywhere.

THEOREM 1. *Multiplication in L_0 is continuous.*

PROOF. This follows, with the aid of the lemma, from the uniform boundedness theorem. It must be verified that if $f_m \in L_0$ and $g_m \in L_0$ and if $f_m \to f \in L_0$ and $g_m \to g \in L_0$ then $f_m g_m \to fg$.

Now, $q_0(f_m g_m - fg) \leqq q_0(f_m g_m - f_m g) + q_0(f_m g - fg)$. From the lemma, it follows that $q_0(f_m g - fg) \to 0$. It remains to show that $q_0(f_m(g_m - g)) \to 0$. But, since $f_m \to f$, it follows that the sequence $f_m h$ is convergent, and therefore bounded, for any $h \in L_0$. The uniform boundedness theorem then states that $q_0(f_n(g_m - g)) \to 0$ uniformly in n, and the theorem follows.

PROPOSITION 5. *The set* $[L_0]^{-1}$ *of invertible elements of the algebra* L_0 *is not open, and inversion is not continuous.*

PROOF. Suppose $f \in [L_0]^{-1}$. Let $E \subset [0, 1]$ be a set of positive measure, less than (say) ϵ. Let $f_\epsilon(x) = f$ if $x \notin E$ and $f_\epsilon(x) = 0$ if $x \in E$. It is clear that f_ϵ is not invertible and that $f_\epsilon \to f$ as $\epsilon \to 0$. By taking f_ϵ small but positive on E, one also sees that inversion is not continuous.

The algebra L_0 is commutative and has a multiplicative identity. By Zorn's lemma, it must have at least one proper maximal ideal. On the other hand, there are no nonzero continuous linear functionals, so there are no closed ideals of codimension 1. This suggests that there may be no proper closed maximal ideals; the following theorem establishes this fact.

THEOREM 2. *The algebra* L_0 *has no proper closed maximal ideals.*

PROOF. Suppose I is a maximal ideal in L_0. Let $E \subset [0, 1]$ have measure between 0 and 1. Let $\mathrm{ind}(x:E)$ denote the indicator of E:

$$\begin{aligned} \mathrm{ind}(x:E) &= 1 \quad \text{if } x \in E, \\ &= 0 \quad \text{if } x \notin E. \end{aligned}$$

Let $\backslash E$ denote the complement of E. Then $\mathrm{ind}(\cdot:E)\,\mathrm{ind}(\cdot:\backslash E) = 0$, so either $\mathrm{ind}(\cdot:E) \in I$ or $\mathrm{ind}(\cdot:\backslash E) \in I$, since L_0/I is a field. Using this fact, one may easily construct a sequence of indicator functions, say h_m, such that $h_m \in I$ and $h_m \to 1$ in L_0. It follows that I is not a proper closed ideal.

It appears that there is no hope for representations of a spectral-theoretic nature for L_0. Moreover, it appears doubtful at this point that any general spectral-theoretic results would be true for continuous linear operators on L_0, from consideration of the standard representation of L_0 as multiplication operators. The operator M_f corresponding to multiplication by a function f (or, equivalently, the function f) may have points in its spectrum. In particular, if $f = \lambda$ on a set of positive measure, then λ belongs to the spectrum of f or M_f.

And, if f has a nonempty spectrum, the spectrum may clearly be unbounded.

A function $f \in L_0$ can have an empty spectrum, however. Consider in particular the function $j(x) = x$. Since $1/(x-a)$ belongs to L_0 for any $a \in [0, 1]$, it follows that $r(\lambda, j) = (\lambda - j)^{-1}$ belongs to L_0 for every complex number λ.

The properties of L_0 established above are also shared by the algebra M of all measurable functions on $[0, 1]$, given the topology of convergence in measure. They are in striking contrast to the properties of locally bounded topological algebras first established by Żelazko [5]. In the next section, some consequences of a property of L_0 not shared by M are considered.

Properties related to subharmonicity. It is easily verified that the function $\log(1+|z|)$ is subharmonic on the entire complex plane. This implies that the quasinorm q_0 is plurisubharmonic on L_0. Recall that a continuous functional ϕ on a complex topological linear space E is plurisubharmonic if for every $x, y \in E$ the function $\lambda \rightarrow \phi(x+\lambda y)$ is subharmonic [3]. This property of L_0 is not shared by M; the topology of the latter can not be defined by a plurisubharmonic quasinorm [2].

The fact that q_0 is plurisubharmonic permits the inference of special properties of analytic functions of a complex variable taking values in L_0. Since the topology of L_0 is neither locally convex nor locally bounded, some care must be exercised in selecting a definition of analyticity. A suitable definition was given in [2]. It is repeated here for the convenience of the reader.

Definition. Let G be a connected open set in the complex plane, and let $f: G \rightarrow L_0$ be a function. Then f is said to be analytic if there exists a sequence of functions $f_m: G \rightarrow L_0$ with the following properties:

(i) The range of f_m is contained in a finite-dimensional subspace E_m of L_0.

(ii) Each f_m is differentiable.

(iii) If cl cvx(A) denotes the closed convex hull of a set $A \subset L_0$ then cl cvx$((f_m - f)[K]) \rightarrow 0$ in the Hausdorff sense for every compact $K \subset G$.

This definition was designed to permit integration, so that the Cauchy integral formula could be established for the above functions. The elements of a theory of such functions were given in [2], from which the following proposition is quoted.

Proposition 6. *Let X and Y be metric linear spaces. Let G be open* *in the complex plane and let* $g:G\to X$ *be analytic. Let* $T:X\to Y$ *be a continuous linear operator. Then* $Tg:G\to Y$ *is analytic.*

This can be applied to the resolvent, to establish that the resolvent cannot be analytic over the entire complex plane.

THEOREM 3. *Let* $f\in L_0$ *be essentially bounded; i.e.,*

$$q_\infty(f) = \text{ess sup}\{ |f(x)|:0 \leqq x \leqq 1 \} < \infty.$$

Then:

(i) $r(\lambda, f) = (\lambda - f)^{-1}$ *is analytic for* $|\lambda| > q_\infty(f)$;

(ii) $r(\lambda, f)$ *is not analytic on the entire complex plane.*

PROOF. Note that the mapping $\lambda\to r(\lambda, f)\in L_\infty$ is analytic if $|\lambda| > q_\infty(f)$. It follows from Propositions 6 and 3 that the mapping $\lambda\to r(\lambda, f)\in L_0$ is also analytic. Suppose $r(\lambda, f)$ is analytic, as an L_0-valued function, for all λ. Then $q_0(r(\lambda, f))$ is subharmonic on the entire complex plane. But $r(\lambda, f)\to 0$ in L_∞ as $\lambda\to\infty$, so $q_0(r(\lambda, f))\to 0$ as $\lambda\to\infty$. Then one would have $q_0(r(\lambda, f))=0$ for all λ; which is impossible.

The proof of Theorem 3 is formally identical to a common proof that the spectrum of an element in a Banach algebra is not empty. The same proof was applied in [2] to verify that the spectrum of a continuous linear operator $T:L_p\to L_p$, for $0<p<1$, is not empty, as another example of the use of the theory developed there. The author was not acquainted with Żelazko's work [5] at the time, and wishes to thank Professor Waelbroeck for calling it to his attention. The lack of a proper attribution of this result to Żelazko in [2] is regretted.

THEOREM 4. *Let* $f\in L_0$; *then* $r(\lambda, f) = (\lambda - f)^{-1}$ *is not analytic over the entire complex plane.*

PROOF. Let $E_m = \{x:f(x)\leqq m\}$. Clearly E_m has positive measure if m is sufficiently large. Let $L_0(E_m)$ denote the space of log-summable functions over the measurable set E_m. The mapping $K_m:L_0\to L_0(E_m)$ defined by $K_m(g) = g|E_m$ (the restriction to E_m) is a continuous algebra homomorphism. Since $K_m(f)$ is bounded on E_m, the resolvent $r_m(\lambda, f) = K_m(r(\lambda, f))$, considered as a function with values in $L_0(E_m)$, is not analytic on all of the complex plane. If $r(\lambda, f)$ were analytic on the entire plane, then $r_m(\lambda, f)$ would also be.

Using the same arguments, it is possible to establish the following slight refinement of these results.

COROLLARY. *Let* $f\in L_0$ *and let* G *be an open connected set in the com-*

plex plane. Let $N_G = \{x : f(x) \in G\}$ *have measure* 0. *Then* $r(\lambda, f)$ *is analytic on G but not on the entire complex plane.*

It may be remarked that if the resolvent set of f were redefined to be the set on which the resolvent $r(\lambda, f)$ is analytic, then much of the standard spectral theory could actually be carried over to this situation, provided the resolvent set is not empty. However, there do exist functions in L_0 having a spectrum, in the ordinary sense, which is dense in the complex plane. Let $\{u_{mk} : k = 1, 2, 3, \cdots\}$ be an enumeration of the rational points in the region $[m \leqq |z| < m+1]$, $m = 0, 1, 2, \cdots$. Define a_{mk} and b_{mk}, for $m = 0, 1, 2, \cdots$ and $k = 1, 2, 3, \cdots$ by:

$$a_{mk} = (1/2)^{m+1} + (1/2)^{m+1+k}, \qquad b_{mk} = (1/2)^{m+1} + (1/2)^{m+k}.$$

Define the function f by $f(x) = u_{mk}$ on the half-open interval $(a_{mk}, b_{mk}]$. It follows that $|f(x)| < m+1$ on the interval $((1/2)^{m+1}, (1/2)^m]$, so $f \in L_0$, and that u_{mk} is in the ordinary spectrum of f.

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INSTITUTE FOR DEFENSE ANALYSES, ARLINGTON, VIRGINIA 22202