A NEW DEFINITION OF A REDUCED FORM

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Introduction. Let a ternary quadratic form $f=ax^2+by^2+cz^2+2ryz+2sxz+2txy$, with integral coefficients, be denoted by $f=[a, b, c, r, s, t]$. Associated with f is the determinant d ($\neq 0$) defined by

$$d = \begin{vmatrix} a & t & s \\ t & b & r \\ s & r & c \end{vmatrix}.$$

We consider in this paper indefinite forms (forms which represent both positive and negative integers) as well as positive forms (those which represent only nonnegative integers). All number symbols which appear in this paper represent integers.

In [3] it is shown that every primitive ternary quadratic form is equivalent to a primitive form in which the coefficients of $2xz$ and $2xy$ are one and zero respectively. In this present paper we find that any ternary quadratic form f_2 is equivalent to a form $f=[a, b, c, r, s, t]$ in which a (or $-a$) is the minimum nonzero integer represented by $|f_2|$ (called the first minimum; b (or $-b$) is the second minimum represented by f_2; c (or $-c$) is the third minimum represented by f_2). (See definitions for second and third minimums.) Finally either $t \mid (|a|, d)$ and $s=0$ or $t=0$ and $s \mid (|a|, d)$. Thus our definition of a reduced form is quite simple. (See definition of a reduced form and conventions adopted to insure uniqueness.)

THEOREM 1. *Let* $f_2=[a_2, b_2, c_2, r_2, s_2, t_2]$ *be a form. Let*

$$m_1 = \min\{\, |f_2(x, y, z)| : f_2(x, y, z) \neq 0\}. \tag{1}$$

Then f_2 *is equivalent to a form* $f_1=[a, b_1, c_1, r_1, s_1, t_1] \ni m_1=|a|$, a *is represented primitively by* f_2 *and* $g_1 \mid g$ *where*

$$g_1 = (a, t_1, s_1) \quad \text{and} \quad g = (a, d). \tag{2}$$

PROOF [1]. Let $a=f_2(k_1, l_1, n_1)$ where $m_1=|f_2(k_1, l_1, n_1)|$. Then $(k_1, l_1, n_1)=1$, for, if $(k_1, l_1, n_1)=g_2>1$, then $f_2(k_1/g_2, l_1/g_2, n_1/g_2)=a/g_2$. But $|a|/g_2<|a|$, contradicting the definition of a. Hence a is represented primitively by f_2. Further if $g_3=(k_1, l_1)$, then $(g_3, n_1)=1$ and there exist integers k_2, l_2, k_3 and $n_3 \ni k_1l_2-l_1k_2=g_3$, and $g_3n_3-n_1k_3=1$. Then the transformation

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$$(3)\qquad T' = \begin{pmatrix} k_1 & k_2 & k_1k_3/g_3 \\ l_1 & l_2 & l_1k_3/g_3 \\ n_1 & 0 & n_3 \end{pmatrix}$$

has determinant $|T'|=1$, since

$$n_3(k_1l_2 - k_2l_1) + n_1k_3/g_3(k_2l_1 - k_1l_2)$$
$$= (g_3n_3 - n_1k_3)(k_1l_2 - k_2l_1)/g_3 = 1.$$

Thus T' applied to f_2 gives an equivalent form. Also, this equivalent form is f_1 having a as the coefficient of x^2.

To show that $g_1|g$, observe that $aA+t_1T_1+s_1S_1=d$, where A, T_1 and S_1 are the cofactors of a, t_1 and s_1 respectively. Thus by (2) $g_1|d$ so that $g_1|g$.

DEFINITIONS. Let Σ be defined by

$$(4)\qquad \Sigma = \{(x, y, z): ax + t_1y + s_1z = 0, g \text{ or } -g\},$$

where g is defined in (2) and a, t_1, and s_1 in Theorem 1. Define a', t' and s' by

$$(5)\qquad a' = a/g_1, \qquad t' = t_1/g_1, \qquad s' = s_1/g_1, \quad \text{where} \quad (a', t', s') = 1,$$

and g_1 is defined in (2).

THEOREM 2. *Let* $f_1=[a, b_1, c_1, r_1, s_1, t_1]$, m, g_1 *be given as in Theorem* 1 *and* a', t', s' *by* (5). *Then* f_1 *is equivalent to* $f=[a, b, c, r, s, t]$ *where*

$$(6)\qquad |t| = g_1 \quad \textit{and} \quad s = 0.$$

PROOF. If $s_1t_1=0$, then replacing x by $x+h_1y_1+h_2z_1$ in f_1 for some h_1 and h_2 gives an equivalent form with the same leading coefficient a and with the coefficients of $2xz_1$ and $2xy_1$ not zero. Thus we may assume that $s_1t_1\neq 0$ in f_1.

We apply the transformation

$$(7)\qquad T = \begin{pmatrix} 1 & \rho & \sigma \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix} \ni |T| = 1$$

to f_1. Then

$$(8)\qquad \begin{aligned} &a = f_1(1, 0, 0), \qquad b = f_1(\rho, \alpha, \gamma), \qquad c = f_1(\sigma, \beta, \delta), \\ &t = a\rho + t_1\alpha + s_1\gamma, \qquad s = a\sigma + t_1\beta + s_1\delta. \end{aligned}$$

We show first that there exists a triple $(x, y, z)=(\rho, \alpha, \gamma)$ satisfying (4) with $t=\pm g_1$ and with $(\alpha, \gamma)=1$. Let

(9) $$k' = t'\alpha + s'\gamma.$$

Then by (5), $(8)_4$ with $t = \pm g_1$

(10) $$k' = \pm 1 - a'\rho.$$

If $d_1 = (t', s')$, then

(11) $$t' = d_1 t'', \quad s' = d_1 s'', \quad \text{where} \quad (t'', s'') = 1.$$

Thus we must show that there exists a triple $(\rho, \alpha, \gamma) \ni$

(12) $$k' = d_1(t''\alpha + s''\gamma) = \pm 1 - a'\rho, \quad \text{with} \quad (\alpha, \gamma) = 1.$$

Since $(a', t', s') = 1$ then

(13) $$(d_1, a') = 1.$$

Hence by (13) there exists an integer $\rho \ni$

(14) $$a'\rho \equiv \pm 1 \pmod{d_1}, \quad \text{i.e.,} \quad \pm 1 - a'\rho = d_1\theta$$

for some integer θ. All solutions of (14) are given by

(15) $$\rho = \rho_0 + d_1\theta' \ni \pm 1 - a'\rho_0 = d_1\theta_0,$$

where ρ_0, θ_0 is any solution of $(14)_2$ and θ' is an arbitrary integer. By (15)

(16) $$\pm 1 - a'(\rho - d_1\theta') = d_1\theta_0 \Rightarrow \pm 1 - a'\rho = d_1(\theta_0 - a'\theta').$$

By $(15)_2$ $(a', \theta_0) = 1$. Thus by Dirichlet's theorem there exists $a\theta' \ni$

(17) $$\theta_0 - a'\theta' = d_2$$

where d_2 is a prime and $\ni$

(18) $$(d_2, a') = 1.$$

By $(11)_2$ there exists integers α and $\gamma \ni$

(19) $$t''\alpha + s''\gamma = d_2$$

and

(20) $$(\alpha, \gamma) = 1.$$

(Suppose a prime $p \mid (\alpha, \gamma)$. Then $p = d_2$. By taking $\phi \ni (\phi, d_2) = 1$, we obtain $\alpha' = \alpha + s''\phi$, $\gamma' = \gamma - t''\phi$ as a solution of (19) with $(\alpha', \gamma') = 1$. For if $(\alpha', \gamma') > 1$, then $(\alpha', \gamma') = d_2$. Thus $d_2 \mid s''\phi$ and $d_2 \mid t''\phi$ which contradicts $(11)_2$ since $(\phi, d_2) = 1$.) Hence by $(16)_2$, (17) and (19) there exists a triple (ρ, α, γ) satisfying $(12)_2$ with $(\alpha, \gamma) = 1$. Thus by (12) we note that

(21) $$(k', a') = 1.$$

In view of $(7)_2$ we want to find δ and $\beta \ni$

$$\alpha\delta - \beta\gamma = 1. \tag{22}$$

All solutions of (22) are given by

$$\delta = \delta_0 - \gamma n, \qquad \beta = \beta_0 - \alpha n, \tag{23}$$

where δ_0, β_0 is any solution of (22) and n is an arbitrary integer. Thus by (5), $(6)_2$, $(8)_5$, (9) and (23)

$$a'\sigma - k'n = s/g_1 - (t'\beta_0 + s'\delta_0), \qquad s = 0. \tag{24}$$

Hence it is seen by (21), that (24) has a solution in σ and n. Any such solution gives a triple $(\sigma, \beta, \delta) \ni$ (24) holds, and $(8)_5$ with $s=0$ also holds. [In fact all solutions of (24) are given by

$$\sigma = \sigma_0 - k'k, \qquad n = n_0 - a'k \tag{25}$$

where σ_0, n_0 is any solution of (24) and k is an arbitrary integer. Thus by (23) and (25) we determine β and δ by

$$\beta = \beta_0 - \alpha n_0 + \alpha a'k, \qquad \delta = \delta_0 - \gamma n_0 + \gamma a'k.] \tag{26}$$

REMARKS. The transformation which replaces y by $-z$ and z by y applied to f of Theorem 2 gives a new form in which a remains unchanged, the coefficient of $2xz$ is $\pm g_1$ and the coefficient of $2xy$ is zero. However we give Theorems 3 and 4 for the purpose of numerical calculation.

THEOREM 3. *Let* $f_1 = [a, b_1, c_1, r_1, s_1, t_1]$, m_1, g_1 *be given as in Theorem* 1; a', t', s' *by* (5). *Let* σ *and* n *satisfy*

$$a[\alpha(\sigma + \rho n) - \rho\beta] = \alpha s - s_1, \qquad \textit{for some } \beta, \tag{27}$$

where (ρ, α, γ) *is some triple satisfying* $(8)_4$ *with* $t=0$, $(\alpha, \gamma) = 1$ *and* $\alpha \neq 0$, *then there exists a transformation* T *in* (7) *which transforms* f_1 *into the equivalent form* $f = [a, b, c, r, s, t]$ *where*

$$t = 0 \quad \textit{and} \quad |s| = g_1. \tag{28}$$

PROOF [2]. Let $(x, y, z) = (\rho, \alpha, \gamma)$ in $(3)_1$. Then all solutions of $(8)_4$ with $t=0$ are given by

$$\rho = t'w - s'v, \qquad \alpha = s'u - a'w, \qquad \gamma = a'v - t'u, \tag{29}$$

where u, v, and w are arbitrary integers. Let

$$d_3 = (a', s'),\ d_4 = (a', t');\ \text{then } (a'/d_3, s'/d_3) = (a'/d_4, t'/d_4) = 1. \tag{30}$$

Then by $(29)_2$ and $(29)_3$

$$\alpha/d_3 = s'/d_3u - a'/d_3w, \qquad \gamma/d_4 = a'/d_4v - t'/d_4u.$$

Let u ($\neq 0$) be any fixed integer $\ni (u, a') = 1$. Then by Dirichlet's theorem there exist integers w and $v \ni$

$$(31) \qquad \alpha = d_3p_1, \qquad \gamma = d_4p_2,$$

where p_1 and p_2 are primes and

$$(32) \qquad (p_1, d_4) = (p_2, p_1, d_3) = 1.$$

But

$$(33) \qquad (d_3, d_4) = 1.$$

For if a prime $p \mid d_3$ and $p \mid d_4$, then $p \mid (a', t', s')$ contradicting $(5)_4$. Thus by (29), (31)–(33), there exist a triple (ρ, α, γ) satisfying $(8)_4$ with $t = 0$ and $(\alpha, \gamma) = 1$. We now seek a solution of (24) with $|s| = g_1$. Since $(\alpha, \gamma) = 1$ and $\alpha \neq 0$, let β, δ satisfy (22). Hence

$$(34) \qquad \alpha\delta = 1 + \beta\gamma.$$

Since $t = 0$, then (10) is replaced by

$$(35) \qquad k' = -a'\rho.$$

Thus by (5), (9), (24), (25), (34) and (35), σ and n must satisfy

$$\begin{aligned} a'\alpha(\sigma + \rho n) &= \alpha s/g_1 - (\alpha t'\beta + \alpha\delta s') = \alpha s/g_1 - (\alpha t'\beta + s' + \beta s'\gamma) \\ &= \alpha s/g_1 - s' - \beta(t'\alpha + s'\gamma) = \alpha s/g_1 - s' + a'\beta\rho. \end{aligned}$$

Since (27) holds, the desired transformation in (7) exists.

THEOREM 4. *Let* $f_1 = [a, b_1, c_1, r_1, s_1, t_1]$, m_1 g_1 *be given as in Theorem* 1 *and* a', t', s' *by* (5). *Let* d_3 *and* d_4 *be given by* (30). *Let* ρ *and* β *satisfy*

$$(36) \quad a'\rho = t/g_1 - s'\gamma \quad \textit{and} \quad t'\beta \equiv s/g_1 \pmod{d_3} \qquad \textit{with } \beta\gamma = -1$$

or ρ *and* δ *satisfy*

$$(37) \quad a'\rho = t/g_1 - t'\alpha \quad \textit{and} \quad s'\delta \equiv s/g_1 \pmod{d_4} \qquad \textit{with } \alpha\delta = 1,$$

where $|t| = g_1$ *and* $s = 0$ *or* $t = 0$ *and* $|s| = g_1$, *then there exists a transformation* (7) *which transforms* f_1 *into the equivalent form* $f = [a, b, c, r, s, t] \ni$

$$(38) \qquad |t| = g_1 \quad \textit{and} \quad s = 0 \quad \textit{or} \quad t = 0 \quad \textit{and} \quad |s| = g_1.$$

PROOF. Suppose (36) holds. Let $\alpha = 0$ and (ρ, β, γ) satisfy (36). Since $\beta\gamma = -1$, $(7)_2$ holds. By (36) we observe that $(8)_4$ is satisfied. By $(36)_2$ let σ and δ satisfy

$$a'\sigma + s'\delta = s/g_1 - t'\beta.$$

Thus $(8)_4$ holds.

Suppose (37) holds. Let $\gamma = 0$ and (ρ, α, δ) satisfy (37). Since $\alpha\delta = 1$, $(7)_2$ holds. By $(37)_1$, we observe that $(8)_4$ is satisfied. By $(37)_2$ let σ and β satisfy $a'\sigma + t'\beta = s/g_1 - s'\delta$. Thus $(8)_5$ holds.

DEFINITIONS. Let $\mathfrak{I} = \{T: |T| = 1$ and T transforms a given form f_2 of Theorem 1 into $f = [a, b, c, r, s, t] \ni a = \pm m_1$ of Theorem 1 and either $t|g$ and $s = 0$ or $t = 0$ and $s|g\}$.

(i) $m_1 = \min\{|f_2(x, y, z)| : f_2(x, y, z) \neq 0\}$ is called the *first minimum* of f_2.

(ii) $m_2 = \min\{|b| : T \in \mathfrak{I}\}$ is called the *second minimum* of f_2.

(iii) Let $\mathfrak{I}_1 = \{T: T \in \mathfrak{I}$ and $|b| = m_2\}$. Then $m_3 = \min\{|c| : T \in \mathfrak{I}_1\}$ is called the *third minimum* of f_2.

THEOREM 5. *Let the form f_1 of Theorem 1 be given in which the coefficient a of x^2 satisfies $a = \pm m_1$. Let $g_1 = (a, t_1, s_1)$. Then f_1 is equivalent to $f = [a, b, c, r, s, t]$ where $b = \pm m_2$, $c = \pm m_3$, and where either $|t| = g_1$ and $s = 0$ or $t = 0$ and $|s| = g_1$.*

PROOF. By Theorem 2 there exist transformations which transform f_1 into forms $f = [a, b, c, r, s, t] \ni a = \pm m_1$ and either $|t| = g_1$ and $s = 0$ or $t = 0$ and $|s| = g_1$. Let $\mathfrak{I}_1$ be those transformations for which $b = \pm m_2$. Then let $T \in \mathfrak{I}_1 \ni c = \pm m_3$. Thus T transforms f_1 into the desired form f.

DEFINITION. A form $f = [a, b, c, r, s, t]$ is said to be *reduced* if $a = \pm m_1$, $b = \pm m_2$, $c = \pm m_3$, $r > 0$ and either

$$t|g \quad \text{and} \quad s = 0 \quad \text{or} \quad s|g \quad \text{and} \quad t = 0,$$

where the following conventions are adopted:

(i) If $f = m_1$ and $f = -m_1$, then we choose $a = m_1$. Having fixed the sign of a, we do likewise for the sign of b if such a choice exists. Next the sign of c is determined as positive if a choice exists.

(ii) We choose the smallest possible divisors $|t|$ and $|s|$ of g, subject to (i).

(iii) If f_2 is equivalent to f where $t|g$ and $s = 0$ and also to f where $s|g$ and $t = 0$, we choose the latter as the reduced form provided (i) and (ii) are satisfied.

REMARKS. If $f = [a, b, c, r, s, 0]$ where $r < 0$ and $s < 0$, then the transformation $x = -x'$, $y = -y'$, $z = z'$ of determinant one transforms f into $[a, b, c, -r, -s, 0]$. Similar statements may be made if $r < 0$ and $s > 0$ or $r > 0$ and $s < 0$. Similar remarks apply when $f = [a, b, c, r, 0, t]$.

COROLLARY. *Let f_1 be given as in Theorem* 1. *If* $g=1$, *then f_1 is equivalent to a reduced form* $[a, b, c, r, s, t]$ *where either* $t=1$ *and* $s=0$ *or* $t=0$ *and* $s=1$.

THEOREM 6. *Any ternary quadratic form f_2 is equivalent to a unique reduced form.*

PROOF. By Theorem 1, f_2 is equivalent to a form $f_1=[a, b_1, c_1, r_1, s_1, t_1]$ in which $|a|=m_1$. Then Theorem 5 gives the desired conclusion since by Theorem 1, $g_1|g$.

EXAMPLE 1. $f_1=3x_1^2+3y_1^2+3z_1^2-2y_1z_1-2x_1z_1-2x_1y_1$ with determinant $d=16$ is a reduced form according to Eisenstein's definition. What is the equivalent reduced form according to our definition?

Now $\phi=3f_1=X^2+2Y^2+6z_1^2$, $X=3x_1-y_1-z_1$, $Y=2y_1-z_1$. Thus $\phi(\pm 1, \pm 1, 0)=3$. But x_1 and y_1 are not integers. Hence $f_1\neq 1$. Likewise $f_1\neq 2$. Thus $f_1(1, 0, 0)=3$ is the first minimum of f_1. Also $f_1(0, 1, 0)=3$ is the second minimum of f_1. Hence by (7)

$$T=\begin{pmatrix}1 & 0 & \sigma\\ 0 & 1 & \beta\\ 0 & 0 & \delta\end{pmatrix} \ni |T|=1$$

where $\rho=0$, $\alpha=1$, $\gamma=0$ with σ, β and δ yet to be determined. Thus $1=|T|=\delta$. By (2) $g_1=(3, -1, -1)=1$. Hence by (6) of Theorem 2, $t=\pm 1$ and $s=0$ since by $(8)_4$ $t=-1$ holds. According to our definition we choose $s=0$ in $(8)_5$. Thus we seek the minimum value of $f_1(\sigma, \beta, 1)$ where σ and β are integral solutions of $3\sigma-\beta-1=0$. Now all solutions of this equation are given by $\sigma=1-n$ and $\beta=2-3n$, where n is an arbitrary integer. Hence $f_1(1-n, 2-3n, 1)=6(2n-1)^2+2\Rightarrow f_1(1, 2, 1)=8$ is a third minimum. Thus applying the transformation $X_1=x+z$, $y_1=y+2z$ and $z_1=z$ we obtain the equivalent form $g=3x^2+3y^2+8z^2+8yz-2xy$. But replacing y by $-y$ and z by $-z$ in g replaces g by the reduced form $f=3x^2+3y^2+8z^2+8yz+2xy$. Note that $f_1(0, -1, 1)=8$. Thus the transformation $x_1=x$, $y_1=y-z$ and $z=z$ applied to f_1 gives the equivalent form $g=3x^2+3y^2+8z^2-8yz-2xy$. Replacing x by $-x$ and z by $-z$ gives f.

EXAMPLE 2. Let $f_2=3x_2^2+7y_2^2+43z_2^2+2y_2z_2+12x_2z_2+8x_2y_2$ with determinant $d=8$. Find the reduced form equivalent to f_2.

By completing the squares in x_2, y_2 and z_2, we find the first minimum of f_2 to be given by $f_2(-7, 4, 1)=2$. Thus in (3) take $k_1=-7$, $l_1=4$ and $n_1=1$. Hence $g_3=(-7, 4)=1$. Thus $l_2=1$, $k_2=-2$ is a solution of $-7l_2-4k_2=1$. In $n_3-k_3=1$, take $n_3=1$ and $k_3=0$. Hence

$$T' = \begin{pmatrix} -7 & -2 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad |T'| = 1,$$

takes f_2 into $f_1 = [2, 3, 43, -11, 5, -1]$. Thus $\phi = 10f_1 = 5X^2 + Y^2 + 16z_1^2$ where $X = 2x - y_1 + 5z_1$ and $Y = 5y_1 - 17z_1$. Hence we find that $X = 0$, $Y = 2$ and $z_1 = -1 \Rightarrow x = 1$, $y_1 = -3$, and $z_1 = -1$ so that $f_1(1, -3, -1) = 2$ is a possible second minimum. By (2), $g = (a, d) = (2, 8) = 2$ and $g_1 = (a, t_1, s_1) = (2, -1, 5) = 1$. From $(8)_2$ and $f_1(1, -3, -1)$ we conclude that $\rho = 1$, $\alpha = -3$, $\gamma = -1$. From (4) and (5) $a'\rho + t'\alpha + s'\gamma = 2(1) + (-1)(-3) + 5(-1) = 0$ so that $f_1(1, -3, -1) = 2$ is a second minimum provided (27) holds; i.e., if $a[\alpha\sigma + (\alpha\rho)n - \rho\beta] = 2[-3\sigma - 3n - \beta] = \alpha s - s_1 = -3s - 5$, where $s = 1$. Thus (27) has a solution in σ and n if we take $\beta = 1$ such that $1 = |T| = \alpha\delta - \beta\gamma = -3\delta + 1$. Thus $\delta = 0$. Now (27) reduces to

$$\sigma + n = 1. \tag{39}$$

All solutions of (39) are given by

$$\sigma = k, \qquad n = 1 - k \tag{40}$$

where k is an arbitrary integer. All solutions of $|T| = 1$, i.e. of $-3\delta + \beta = 1$ are given by

$$\delta = n, \qquad \beta = 1 + 3n \tag{41}$$

where n is an arbitrary integer. By (40) and (41)

$$\sigma = k, \qquad \beta = 4 - 3k, \qquad \delta = 1 - k, \tag{42}$$

where k is to be chosen so that $f_1(k, 4-3k, 1-k)$ is a third minimum. Now $f_1(k, 4-3k, 1-k) = 2k^2 - 2k + 3 \Rightarrow 2f_1 = (2k-1)^2 + 5$. Take $k = 1$ and hence by (42), $f(1, 1, 0) = 3$. By (7)

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

takes f_1 into the reduced form $f = 2x^2 + 2y^2 + 3z^2 + 2yz + 2xz$. If we had chosen $k = 0$, then T with its 3rd column replaced by $(0, 4, 1)$ takes f_1 into $[2, 2, 3, -1, 1, 0]$ which is equivalent to f by replacing x by $-x$ and z by $-z$.

EXAMPLE 3. Let $f_2 = 2yz + 2xz + 2xy$ with determinant $d = 2$. Find the reduced form f equivalent to f_2.

$$m_1 = a = f_2(1, 1, 0) = 2.$$

By Theorem 1, we may take

$$T' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus T' transforms f_2 into $f_1 = [2, 4, 0, 3, 2, 3]$. By (2) $g = (2, 2) = 2$ and $g_1 = (2, 3, 2) = 1$. By (8) and Theorems 4 and 5, we choose ρ, α and γ such that $|f_1(\rho, \alpha, \gamma)| = |b| = m_2$, a second minimum, and where by $(8)_4$ $2\rho + 3\alpha + 2\gamma = 0$. Now $2f_1 = X^2 - Y^2 - 4z^2$ where $X = 2x + 3y + 2z$, and $y = Y$. Thus we take $x = \rho$, $y = \alpha$, and $z = \gamma$ with $X = Y = 0$ and $z = -1$ so that $f_1 = -2 = b$. Then $2\rho + 3\alpha = 2$ holds with $\alpha = 0$ and $\rho = 1$. Note that $\gamma = z = -1$. Thus (36) holds for $2(1) = 0/1 - 2(-1)$, $\beta\gamma = -1 \Rightarrow \beta = 1$ and $3(1) \equiv 1/1 (\bmod d_3)$ where, by (30), $d_3 = (2, 2) = 2$ and s is taken $= g_1 = 1$. It remains to choose σ and δ in (7) so that $|c| = |f_1(\sigma, 1, \delta)| = m_3$ and such that $(8)_5$ holds, i.e., $2\sigma + 3(1) + 2\delta = s$ where $s = 1$. Thus we choose σ and δ so that $\sigma = -(\delta + 1)$ and $m_3 = |f_1[-(\delta+1), 1, \delta]| = |-2\delta^2| \geqq m_2 = 2$. Hence take $\delta = -1$ so that $\sigma = 0$. Thus by (7)

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

transforms f_1 into $f = [2, -2, -2, -2, 1, 0]$ with determinant $d = 2$ which is equivalent to the reduced form $[2, -2, -2, 2, 1, 0]$.

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