A NOTE ON FINITE METABELIAN $p$-GROUPS

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Abstract. Let $A$ be an abelian subgroup of maximal order in the finite metabelian $p$-group $P$. It is shown that there exists a normal abelian subgroup $A_1$ of $P$ such that the order of $A_1$ is equal to the order of $A$.

In [1], J. L. Alperin raised the following question. If $p$ is any prime and $A$ is an abelian subgroup of index $p^n$ in the finite $p$-group $P$, does there exist a normal abelian subgroup of $P$ of index $p^n$? Alperin has shown in [1] that the answer to this question is yes, if $n$ is 2 or 3. The purpose of this note is to answer this question in the affirmative in the special case that $P$ is metabelian. As in [2], for any $p$-group $P$ we let $A(P)$ be the set of abelian subgroups of $P$ of maximal order. We shall prove that if $P$ is metabelian there exists a normal subgroup $A$ belonging to $A(P)$. For metabelian $p$-groups, this clearly implies an affirmative answer to Alperin's question. The notations and terminology are standard.

Lemma 1. Suppose $P$ is a finite metabelian $p$-group and $A$ belongs to $A(P)$. Then the following are true.

(i) If $x \in P$, $[x, A] = [x, A] \in A(P)$.
(ii) $[x, y, z] [z, x, y] [y, z, x] = 1$ for all $x, y, z$ in $P$.
(iii) If $x \in P$, the order of $A/C_A[x, A]$ is equal to the order of $[x, A]/A \cap [x, A]$.
(iv) If $x \in P$, $C_A[x, A] = A \cap A^x$ and $C_A[\langle x \rangle, A] = \cap (A^y : y \in \langle x \rangle)$.

Proof. Statement (i) is contained in [2, Theorem 2.4, p. 272]. Statement (ii) is well known and follows directly from [2, Theorem 2.3, p. 19]. The third statement follows from (i), since the order of $A$ is equal to the order of $[x, A]C_A[x, A]$. For (iv), $a \in C_A[x, A]$ if and only if $a \in A \cap C_A((b^{-1})^x)$ for all $b \in A$, if and only if $a \in A \cap C_P(A^x) = A \cap A^x$. The second part of (iv) follows in exactly the same way.

Lemma 2. Suppose $P$ is a finite metabelian $p$-group, $A \in A(P)$, and $x \in P$ is such that $[x, A] \leq N_P(A)$. Then $C_A[x, A] = C_A[\langle x \rangle, A]$, and so $[\langle x \rangle, A]C_A[\langle x \rangle, A] \in A(P)$.

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1 After submitting this note, the author became aware that J. L. Alperin has proved the result, as well as some stronger results.
Proof. Define the map $f: A \to [x, A] / A \cap [x, A]$ by $f(a) = [x, a] \pmod{A \cap [x, A]}$ for all $a \in A$. If $a_1, a_2 \in A$, then $[x, a_1 a_2] = [x, a_2] \cdot [x, a_1]$ and $[x, a_1, a_2] \in A \cap [x, A]$ by hypothesis. Since $P$ is metabelian, $f$ is a homorphism, and $f$ is clearly an onto map. If $a \in \text{Ker}(f)$, then $[x, a] \in A$, and so $(a^{-1})^z \in A$. Hence $a \in A \cap A^{-1}$. Also since $[x, a] \in A$ we have $[x, a, a_1] = 1$ for all $a_1$ in $A$. By [2, Lemma 2.5, p. 20] we obtain $[x, a_1, a_2] = 1$ for all $a_2$ in $A$, and therefore $a \in C_A[x, A] = A \cap A^z$. Therefore $\text{Ker}(f) \leq A \cap A^z \cap A^{-1} \leq A \cap A^z$, but by Lemma 1 we must have equalities. It now follows easily that $C_A[x, A] = A \cap A^z = \cap A^y : y \in \langle x \rangle = C_A[\langle x \rangle, A]$. Since $[\langle x \rangle, A] \geq [x, A], [\langle x \rangle, A] C_A[\langle x \rangle, A] \in A(P)$.

Theorem. If $P$ is a finite metabelian $p$-group, then there exists $A \in A(P)$ such that $A$ is normal in $P$.

Proof. Let $A \in A(P)$ and $M$ be a maximal subgroup of $P$ containing $A$. Inductively we may assume $A$ is normal in $M$. Choose $x \in P$ such that $P = \langle x, M \rangle$. Then $[x, A] \leq M \leq N_P(A)$, and by Lemma 2, $A_1 = \langle x \rangle, A \mid C_A[\langle x \rangle, A] \in A(P)$. We now show that $A_1$ is normal in $P$. Trivially $x \in N_P[\langle x \rangle, A]$. Let $m \in M$ and $[x^i, a]$ be a generator of $\langle x \rangle, A$. Then $[x^i, a]^m = [x^i, a]\cdot [x^i, a, m]$ and by Lemma 1, $[x^i, a, m] = [a, m, x^i]^{-1} [m, x^i, a]^{-1}$ which belongs to $A_1$. Let $a_1 \in C_A[\langle x \rangle, A]$, then $a_1^z = a_1 [a_1, x] \in A_1$; if $m \in M$, then $[a_1^m, [\langle x \rangle, A]] = [a_1, [\langle x \rangle, A]]^m = [a_1, A_1]^m = 1$. Therefore $P = \langle x, M \rangle$ normalizes $A_1$.

References


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