A NOTE ON FINITE METABELIAN p-GROUPS[1]

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ABSTRACT. Let A be an abelian subgroup of maximal order in the finite metabelian p-group P. It is shown that there exists a normal abelian subgroup A_1 of P such that the order of A_1 is equal to the order of A.

In [1], J. L. Alperin raised the following question. If p is any prime and A is an abelian subgroup of index p^n in the finite p-group P, does there exist a normal abelian subgroup of P of index p^n? Alperin has shown in [1] that the answer to this question is yes, if n is 2 or 3. The purpose of this note is to answer this question in the affirmative in the special case that P is metabelian. As in [2], for any p-group P we let $A(P)$ be the set of abelian subgroups of P of maximal order. We shall prove that if P is metabelian there exists a normal subgroup A belonging to $A(P)$. For metabelian p-groups, this clearly implies an affirmative answer to Alperin's question. The notations and terminology are standard.

LEMMA 1. *Suppose P is a finite metabelian p-group and A belongs to $A(P)$. Then the following are true.*

(i) *If* $x\in P$, $[x, A]C_A[x, A]\in A(P)$.

(ii) $[x, y, z][z, x, y][y, z, x]=1$ *for all* x, y, z *in* P.

(iii) *If* $x\in P$, *the order of* $A/C_A[x, A]$ *is equal to the order of* $[x, A]/A\cap[x, A]$.

(iv) *If* $x\in P$, $C_A[x, A]=A\cap A^x$ *and* $C_A[\langle x\rangle, A]=\bigcap(A^y : y\in\langle x\rangle)$.

PROOF. Statement (i) is contained in [2, Theorem 2.4, p. 272]. Statement (ii) is well known and follows directly from [2, Theorem 2.3, p. 19]. The third statement follows from (i), since the order of A is equal to the order of $[x, A]C_A[x, A]$. For (iv), $a\in C_A[x, A]$ if and only if $a\in A\cap C_A((b^{-1})^x)$ for all $b\in A$, if and only if $a\in A\cap C_P(A^x)$ $=A\cap A^x$. The second part of (iv) follows in exactly the same way.

LEMMA 2. *Suppose P is a finite metabelian p-group, $A\in A(P)$, and $x\in P$ is such that $[x, A]\leq N_P(A)$. Then $C_A[x, A]=C_A[\langle x\rangle, A]$, and so $[\langle x\rangle, A]C_A[\langle x\rangle, A]\in A(P)$.*

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[1] After submitting this note, the author became aware that J. L. Alperin has proved the result, as well as some stronger results.

PROOF. Define the map $f: A \to [x, A]/A \cap [x, A]$ by $f(a) \equiv [x, a]$ (mod $A \cap [x, A]$) for all $a \in A$. If $a_1, a_2 \in A$, then $[x, a_1a_2] = [x, a_2] \cdot [x, a_1][x, a_1, a_2]$ and $[x, a_1, a_2] \in A \cap [x, A]$ by hypothesis. Since P is metabelian, f is a homorphism, and f is clearly an onto map. If $a \in \mathrm{Ker}(f)$, then $[x, a] \in A$, and so $(a^{-1})^x \in A$. Hence $a \in A \cap A^{x^{-1}}$. Also since $[x, a] \in A$ we have $[x, a, a_1] = 1$ for all a_1 in A. By [2, Lemma 2.5, p. 20] we obtain $[x, a_1, a] = 1$ for all a_1 in A, and therefore $a \in C_A[x, A] = A \cap A^x$. Therefore $\mathrm{Ker}(f) \leqq A \cap A^x \cap A^{x^{-1}} \leqq A \cap A^x$, but by Lemma 1 we must have equalities. It now follows easily that $C_A[x, A] = A \cap A^x = \bigcap(A^y : y \in \langle x \rangle) = C_A[\langle x \rangle, A]$. Since $[\langle x \rangle, A] \geqq [x, A]$, $[\langle x \rangle, A]C_A[\langle x \rangle, A] \in A(P)$.

THEOREM. *If P is a finite metabelian p-group, then there exists $A \in A(P)$ such that A is normal in P.*

PROOF. Let $A \in A(P)$ and M be a maximal subgroup of P containing A. Inductively we may assume A is normal in M. Choose $x \in P$ such that $P = \langle x, M \rangle$. Then $[x, A] \leqq M \leqq N_P(A)$, and by Lemma 2, $A_1 = [\langle x \rangle, A]C_A[\langle x \rangle, A] \in A(P)$. We now show that A_1 is normal in P. Trivially $x \in N_P[\langle x \rangle, A]$. Let $m \in M$ and $[x^i, a]$ be a generator of $[\langle x \rangle, A]$. Then $[x^i, a]^m = [x^i, a][x^i, a, m]$ and by Lemma 1, $[x^i, a, m] = [a, m, x^i]^{-1}[m, x^i, a]^{-1}$ which belongs to A_1. Let $a_1 \in C_A[\langle x \rangle, A]$, then $a_1^x = a_1[a_1, x] \in A_1$; if $m \in M$, then $[a_1^m, [\langle x \rangle, A]] = [a_1, [\langle x \rangle, A]^{m^{-1}}]^m \leqq [a_1, A_1]^m = \langle 1 \rangle$. Therefore $P = \langle x, M \rangle$ normalizes A_1.

REFERENCES

1. J. L. Alperin, *Large Abelian subgroups of p-groups*, Trans. Amer. Math. Soc. **117** (1965), 10–20. MR **30** #1180.

2. D. Gorenstein, *Finite groups*, Harper & Row, New York, 1968. MR **38** #229.

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