DECOMPOSITION NUMBERS OF p-SOLVABLE GROUPS[1]

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ABSTRACT. In the character theory of finite groups one decomposes each ordinary irreducible character χ_i of a group into an integral linear combination of p-modular irreducible characters ϕ_j, $\chi_i = \sum d_{ij}\phi_j$. The nonnegative integers d_{ij} are called the p-decomposition numbers. Let G be a p-solvable group whose p-Sylow subgroups are abelian. If $G/O_{p'p}(G)$ is cyclic the p-decomposition numbers are $\leqq 1$. This condition is far from necessary as any group G with abelian, normal p-Sylow subgroup P with G/P abelian has p-decomposition numbers $\leqq 1$. A result of Brauer and Nesbitt together with the first result yields the following. A group G has a normal p-complement and abelian p-Sylow subgroups if and only if each irreducible character of G is irreducible as a p-modular character.

A group is said to be p-solvable if each of its composition factors is either a p-group or a p'-group, p a prime number. Denote by $O_{p'}(G)$ and $O_{p'p}(G)$ the largest normal p'-subgroup of a group G and the inverse image of the largest normal p-subgroup of $G/O_{p'}(G)$ under the natural homomorphism $G \rightarrow G/O_{p'}(G)$ respectively. The following proposition is 1.2.3 of [6]. If G is a p-solvable group and P is a p-Sylow subgroup of G then the center of P, $Z(P) \subset O_{p'p}(G)$. As a result, if P is abelian $P \subset Z(P) \subset O_{p'p}(G)$ and so $O_{p'p}(G) = PO_{p'}(G)$. It also follows that $[G: O_{p'p}(G)]$ is relatively prime to p.

We assume knowledge of elementary character theory and recall some facts about the relationship between the ordinary and modular representations of a group G. References [1] and [2] will supply the details. Let K be the extension of the rational numbers obtained by adjoining the mth roots of 1 where m is the exponent of G. R denotes the valuation ring of an extension of the p-adic valuation to K in K, $\mathcal{P}$ denotes the maximal ideal of R and $\bar{K} = R/\mathcal{P}$ a field of characteristic p.

If $\mathcal{Z}$ is a K-representation of G, there exists a K-similar representation $\mathcal{Z}'$ such that all the entries of $\mathcal{Z}'(g)$, $g \in G$, lie in R. By reducing the entries of $\mathcal{Z}'(g)$ modulo $\mathcal{P}$ we get a $\bar{K}$-representation $\bar{\mathcal{Z}}$ of G. This process does not determine $\bar{\mathcal{Z}}$ uniquely, but the composition factors of $\bar{\mathcal{Z}}$ are uniquely determined by $\mathcal{Z}$.

Presented to the Society, August 29, 1969; received by the editors September 15, 1969.

AMS Subject Classifications. Primary 2080; Secondary 2027.

Key Words and Phrases. p-solvable group, decomposition numbers, p-modular character, ordinary character.

[1] This research was supported in part by NSF Grants GP 8457 and GP 11542.

Given an irreducible $\bar{K}$-representation F of G, one constructs a function ϕ on the p-regular elements of G as follows. If $g \in G$ is p-regular then the eigenvalues $\{\bar{\delta}_i\}$ of $F(g)$ are m' roots of 1 where $(m', p)=1$ and $m=m'p^n$. Let $\{\delta_i\}$ be the corresponding set of roots of unity in K under the natural homomorphism $R \to \bar{K}$. Define $\phi(g) = \sum \delta_i$. $F \leftrightarrow \phi$ is a one-to-one correspondence between the isomorphism classes of irreducible $\bar{K}$-representations $\{F_i\}$ and a set of functions $\{\phi_i\}$ called the p-modular characters of G. The $\{\phi_i\}$ form a basis of the K vector space of class functions from the p-regular elements of G into K.

It follows from these remarks that if Z_i is an irredubible K-representation of G with character χ_i, then F_j appears as a composition factor of $\bar{Z}_i$ with multiplicity d_{ij} where $\chi_{i|p\text{-regular elements}} = \sum d_{ij}\phi_j$. The matrix $D=(d_{ij})$ is called the decomposition matrix and the non-negative integers d_{ij} are called the decomposition numbers.

It is sometimes convenient to partition the ordinary and modular characters into subsets called blocks, and when the χ's and ϕ's are indexed accordingly the decomposition matrix has block diagonal form. A submatrix on the diagonal of D which corresponds to a block of ordinary and modular characters is called the decomposition matrix of that block.

THEOREM. *Let G be a p-solvable group whose p-Sylow subgroups are abelian. If $G/O_{p'p}(G)$ is cyclic then the p-decomposition numbers are* $\leqq 1$.

The proof is given in three steps. First, using an induction method of Fong [4] we reduce the proof to a consideration of a simpler type of group. Next we reduce the proof to a question about the ordinary characters of this type of group. Last, we use a theorem of Gallagher [5] to deal with these characters.

First observe that the hypotheses of the theorem hold for subgroups H of G which contain $O_{p'}(G)$. Clearly a p-Sylow subgroup of H is abelian. Since $O_{p'}(G) \subset H$, $O_{p'}(G) \subset O_{p'}(H)$. We observed in the introduction that $O_{p'p}(G) = PO_{p'}(G)$ where P is a p-Sylow subgroup of G. Hence

$$\begin{aligned} O_{p'p}(G) \cap H &= O_{p'}(G)P \cap H \subset O_{p'}(H)\, P \cap H \\ &= O_{p'}(H)(P \cap H) \subset O_{p'p}(H). \end{aligned}$$

Thus $H/O_{p'p}(H)$ is a homomorphic image of $H/O_{p'p}(G) \cap H \cong HO_{p'p}(G)/O_{p'p}(G) \subsetneqq G/O_{p'p}(G)$ which is cyclic. Thus $H/O_{p'p}(H)$ is cyclic.

We use induction on $|G|$ to reduce our considerations to groups with a normal p-Sylow subgroup. If $O_{p'}(G)=1$ then G has a normal

p-Sylow subgroup and a cyclic complement. If $O_{p'}(G)\neq 1$ and B is a block of G, then Fong's theorem 2B [4] asserts that there is a subgroup H, $O_{p'}(G)\subset H\subset G$, and a block B' of H whose decomposition matrix is identical to that of B. If $H\neq G$ then all the decomposition numbers for characters in B are $\leqq 1$ by induction. If $H=G$, then Fong's theorem 2D gives a group M with a cyclic, central p'-subgroup C and a block B'' of M such that $M/C\cong G/O_{p'}(G)$ and the decomposition matrix of B'' is the same as that of B. Clearly M is p-solvable with an abelian p-Sylow subgroup P'. Thus $O_{p'p}(M)=CP'=C\times P'$, as C is central, and so $P'\lhd M$. Moreover since $G/O_{p'p}(G)$ is cyclic it follows that $M/P'C_G(P')$ is cyclic. Thus it suffices to prove the theorem for a group G with normal abelian p-Sylow subgroup P and with $G/PC_G(P)$ cyclic.

We isolate the next part of the proof as a

LEMMA. *Suppose $G=PH$ is a finite group, $P\lhd G$ is the p-Sylow subgroup and $(|H|, p)=1$. If ϕ_j is a p-modular irreducible character of G, then $\phi_{j|H}$ is an ordinary irreducible character of H. Hence if χ_i is an ordinary character of G, the equation $\chi_i=\sum d_{ij}\phi_j$ gives a decomposition of $\chi_{i|H}$ as a sum of irreducible characters of H.*

PROOF. Let F be an irreducible $\overline{K}$-representation of G with modular character ϕ. By Clifford's theorem [2, 49.7] $P\subset$kernel of F and so F (ϕ) can be viewed as a representation (modular character) of $G/P\cong H$. But since $p\nmid |H|$ there is a K-irreducible representation Z of H such that $\overline{Z}=F_{|H}$, [3, 4.4]. $\phi_{|H}$ must be the character of Z. The rest follows immediately.

Suppose that G has a normal abelian p-Sylow subgroup P, p-complement H and that $G/PC_G(P)$ is cyclic. By the lemma we must show that $\chi_{|H}$ is multiplicity free for every irreducible character χ of G, i.e. if $\chi_{|H}=\sum a_i\lambda_i$, λ_i irreducible characters of H, then $a_i\leqq 1$ for all i.

By a theorem of Gallagher [5, Theorem 7] every irreducible character of G can be realized in the following way. Let θ be a character of P. Let $T=\{x\in G:\theta^x=\theta\}$ be the stability group of θ and θ_1 an extension of θ to T. Let ω be a character of T with P in its kernel. Then $(\theta_1\omega)^G$, the induced character, is irreducible and all characters of G are of this form.

Observe that $PC_G(P)\subset T$. Since $HP=G$, the only (H, T) double coset is $HT=G$. Thus the subgroup theorem [2, 44.2] implies that $(\theta_1\omega)^G_{|H}=((\theta_1\omega)_{H\cap T})^H$.

Now $H\cap T\lhd H$ as $C_H(P)\subset H\cap T$ and $H/C_H(P)\cong H/H\cap C_G(P)\cong HC_G(P)/C_G(P)=G/C_G(P)$ is cyclic. It also follows that $H/H\cap T$ is cyclic.

Suppose that $((\theta_1\omega)_{H\cap T})^H = a_1\lambda_1 + \cdots + a_n\lambda_n$, λ_i irreducible characters of H. Clifford's theorem and Frobenius reciprocity imply that $\lambda_{i|H\cap T} = a_i \sum_x (\theta_1\omega)^x_{|H\cap T}$ where x runs over a set of coset representatives of $H\cap T$ in the stability group of $\theta_1\omega$ in H. But this stability group is cyclic modulo $H\cap T$ since $H/H\cap T$ is cyclic. Thus a theorem of Schur [**3**, 9.12] implies that $a_i = 1$. This completes the proof.

This theorem is not exhaustive. In fact the following proposition covers many p-solvable groups with abelian p-Sylow subgroups which are not covered by the theorem.

PROPOSITION. *If G is a finite group with normal abelian p-Sylow subgroup and abelian p-complement H, then all p-decomposition numbers are* $\leqq 1$.

PROOF. Let χ be an irreducible character of G. By the lemma we must show that $\chi_{|H}$ is multiplicity free. Now by Fröbenius reciprocity the induced character $\theta^G = \chi + \chi'$ for some (reducible) character χ' of G and some irreducible character θ of P. Thus it suffices to show that $\theta^G_{|H}$ is multiplicity free for every irreducible character θ on P.

$P \lhd G$ implies that $\theta^G(x) = 0$ for $x \in H$, and $\theta^G(1) = |G:P| = |H|$ as θ is linear. Therefore $\theta^G_{|H}$ is the regular character on H, and the regular character of an abelian group contains each of its irreducible constituents with multiplicity 1. This proves the proposition.

On the other hand examples show that none of the hypotheses of the theorem is superfluous. In particular the hypothesis that $G/O_{p'p}(G)$ is cyclic cannot be deleted as the following shows. Let P be an abelian group of type (3, 3) and let $Q = \langle \alpha, \beta \rangle$ be the quaternion group of order 8. Let $Q \to \mathrm{Aut}(P)$ be a homomorphism such that α inverts the elements of P and β exchanges the elements of some basis of P. (Thus the involution $u \in Q$ acts trivially.) Let G be the resulting semidirect product, and identify P and Q with subgroups of G. Let λ be a character on $P \times \langle u \rangle$ defined as the product of the nonidentity character on $\langle u \rangle$ with a linear character on P whose kernel is not fixed by β. One easily checks that $P \times \langle u \rangle$ is the stability group of λ in G and that $P \times \langle u \rangle \lhd G$. Thus λ^G is irreducible. But u is central in G and so $\lambda^G(u) = -4$. Thus $\lambda^G_{|Q} = 2$ times the nonlinear character of Q. Therefore $\lambda^G_{|Q}$ is not multiplicity free.

A result of Brauer, Nesbitt and Osima [**1**], [**7**] together with the theorem yield the following

COROLLARY. *A group G has a normal p-complement and abelian p-Sylow subgroups if and only if each irreducible character restricted to the p-regular elements is a modular irreducible character.*

PROOF. The theorem of Brauer, Nesbitt and Osima says that a group has a normal p-complement if and only if each block of G contains exactly one modular character, i.e. if and only if each row of D has only one nonzero entry. Thus if G has a normal p-complement and abelian p-Sylow subgroups then the nonzero entry in each row must be a 1 by the theorem. Thus each irreducible character is irreducible as a modular character.

Conversely if each character of G is irreducible as a modular character then each row of D consists of 0's and one 1. Thus G has a normal p-complement. If a p-Sylow subgroup P were not abelian then by taking a nonlinear irreducible character of P and extending it to a character χ of G we would have an irreducible character of G which when viewed as a modular character would decompose into $\chi(1)$ copies of the principal modular character contrary to hypothesis. Thus P is abelian.

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