OVERCONVERGENCE AND $(C, 1)$ SUMMABILITY

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ABSTRACT. It is shown that a power series can be diluted to become $(C, 1)$ summable outside the radius of convergence if and only if it overconverges.

Let

$$\sum_{n=0}^{\infty} a_n z^n \tag{1}$$

be a power series with radius of convergence 1. It clearly cannot be $(C, 1)$ summable for any $|z|>1$, since for such z, $a_n z^n \neq o(n)$. Sometimes however it can be diluted (i.e., zeros inserted between the terms) to become $(C, 1)$ summable. In this paper we show that this happens if and only if the series overconverges. In what follows $s_p(z)$ will always denote the pth partial sum of (1). If U is an open set and $F_n(z)$ is a sequence of functions on U converging uniformly on every compact subset of U, then we say F_n converges almost uniformly (a.u.) on U. The series (1) is said to overconverge a.u. on U, if there exists a subsequence $s_{p_n}(z)$ of partial sums converging a.u. on U. A series $\sum f_n(z)$ is said to be $(C, 1)$ summable a.u. on U, if the sequence of $(C, 1)$ means of the partial sums is a.u. convergent on U. A diluted sequence is a sequence obtained by repeating the terms; if we dilute a series $\sum \alpha_n$ by inserting ϵ_n zeros between α_n and α_{n+1}, then the sequence of partial sums gets diluted by repeating the nth partial sum ϵ_n+1 times. If a sequence of functions is a.u. convergent then it remains so after we dilute it. The sequence of $(C, 1)$ means of an a.u. convergent sequence is a.u. convergent.

THEOREM. *Let U be an open neighborhood of a point $|z_1|>1$. Then the series* (1) *can be diluted to become* $(C, 1)$ *summable a.u. in U if and only if it overconverges a.u. on U.*

PROOF. Assume (1) overconverges. Let $s_{n_k}(z)$ be a.u. convergent on U; for simplicity assume $n_1 \geqq 1$. Let K_n be an increasing sequence of

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compact sets whose union is U. Put

$$M(k,j) = \max_{1 \leqq i \leqq j} \sup\{ | s_i(z) | : z \in K_k \}.$$

We now dilute (1) by inserting ϵ_k zeros immediately after $a_{n_k}z^{n_k}$; ϵ_k to be chosen in a moment. This dilutes the sequence of partial sums by having s_{n_k} repeated ϵ_k+1 times. The sequence $\{\sigma_n(z)\}$ of $(C, 1)$ means of the partial sums of the diluted series will have the following form: If $n=\epsilon_1+\epsilon_2+\cdots+\epsilon_k+n_k+m$ $(0 \leqq m \leqq n_{k+1}-n_k)$

$$\begin{aligned} \sigma_n &= n^{-1}(s_1 + s_2 \cdots + s_{n_k+m}) - n^{-1}(n_k + m)s_{n_{k+1}} \\ &\quad + n^{-1}(\epsilon_1 s_{n_1} + \epsilon_2 s_{n_2} + \cdots + (n - \epsilon_1 - \cdots - \epsilon_k)s_{n_{k+1}}). \end{aligned} \tag{2}$$

If $n=\epsilon_1+\cdots+\epsilon_k+n_{k+1}+m$ $(0<m\leqq \epsilon_{k+1})$

$$\begin{aligned} \sigma_n &= n^{-1}(s_1 + \cdots s_{n_{k+1}}) + n^{-1}n_{k+1}s_{n_{k+1}} \\ &\quad + n^{-1}(\epsilon_1 s_{n_1} + \cdots + (n - \epsilon_1 - \cdots - \epsilon_k)s_{n_{k+1}}). \end{aligned} \tag{3}$$

The last terms on the right-hand side of (2) and (3) are $(C, 1)$ means of the diluted sequence $\{s_{n_1}\ \epsilon_1 \text{ times}, s_{n_1}\ \epsilon_2 \text{ times} \cdots\}$, therefore, they are a.u. convergent for any choice of ϵ's. Select now ϵ_k such that $n_{k+1}M(k, n_{k+1})\epsilon_k^{-1} \to 0$ as $k \to \infty$. If $z \in K_p$, then for $k \geqq p$, the remaining terms in (2) and (3) are bounded by $\epsilon_k^{-1}n_{k+1}M(k, n_{k+1})$ uniformly in K_p. Hence σ_n converges a.u. Assume next (1) can be diluted to become $(C, 1)$ summable a.u. in U. Let ϵ_n be the number of zeros inserted after $a_n z^n$ and set $\delta_n=\epsilon_n+1$. Then by hypothesis

$$\sigma_k = (\delta_0 + \cdots + \delta_k)^{-1}(\delta_0 s_0 + \cdots + \delta_k s_k)$$

converges a.u. on U. Since $b_n=o(n)$ for any $(C, 1)$ summable series $\sum b_n$ [1, p. 101], we have for every $z_0 \in U$

$$(\delta_0 + \delta_1 + \cdots + \delta_k)^{-1} a_k z_0^k = o(1)$$

which implies

$$| a_k z_0^k | \leqq N(\delta_0 + \cdots + \delta_k); \quad N\text{-constant}. \tag{4}$$

We want to show that $D_k=(\delta_0+\cdots+\delta_k)\delta_k^{-1}$ has a bounded subsequence. If not, then for any $\epsilon>0$

$$\delta_k < \epsilon(\delta_0 + \cdots + \delta_k) = \epsilon t_k \qquad (k \geqq k_0).$$

But then

$$t_k - t_{k-1} = \delta_k < \epsilon t_k; \qquad t_k < (1-\epsilon)^{-1}t_{k-1} \qquad (k \geqq k_0)$$

so that $t_k < t_{k_0}(1-\epsilon)^{k_0-k}$. For sufficiently small ϵ and $|z_0|>1$ this contradicts (4). Let then D_{k_j} be a bounded subsequence of D_k. We have

$$(\delta_0 + \cdots + \delta_k)\sigma_k - \delta_k s_k = \delta_0 s_0 + \cdots + \delta_{k-1}s_{k-1}$$

or dividing by δ_k, $D_k\sigma_k - s_k = (D_k - 1)\sigma_{k-1}$. So for $k > 1$

$$s_k = D_k(\sigma_k - \sigma_{k-1}) + \sigma_{k-1}. \tag{5}$$

Since D_{k_j} is bounded and σ_k converges a.u., we conclude from (5) that s_{k_j} converges a.u.

The above technique was motivated in part by a paper of S. Mazur [2] who considered rearrangements of $(C, 1)$ summable series.

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References

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