APPLICATIONS OF STEREOGRAPHIC PROJECTIONS TO SUBMANIFOLDS IN E^m AND S^m

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ABSTRACT. In this paper we give a criterion for a compact minimal submanifold of S^m to lie in a given great hypersphere in terms of an integral over the stereographic image in E^m of the submanifold. We also show that if all the points a certain normal distance C from a compact hypersurface M in E^m lie on a sphere of radius $D<C$ then M is a hypersphere. This generalizes a classical result on parallel hypersurfaces. We prove this theorem by showing it to be equivalent, via stereographic projection, to a recent result of Nomizu and Smyth concerning the gauss map for hypersurfaces of S^m.

Introduction. It is well known that the differential geometry of Euclidean space is closely related, via stereographic projection, to the differential geometry of the standard unit sphere. In this paper we study certain simple aspects of this relation.

In §1 we recall some of the basic formulas relating the geometry of the sphere to that of Euclidean space.

In §2 we use several of these formulas, together with several well known integral formulas, to derive a criterion for telling whether a compact minimal submanifold of the sphere lies within a given hypersphere.

In §3 we show that a recent result of Nomizu and Smyth [2] which considers the gauss map for hypersurfaces of spheres is equivalent to an extension of a classical theorem concerning parallel hypersurfaces in Euclidean space.

NOTATION. E^{m+1} is the Euclidean space of dimension $(m+1)$, which we also view in the usual manner as being a vector space with inner product $(\ ,\ )$ and norm $|\ \ |$, S^m denotes the unit sphere in E^{m+1}, consisting of all unit vectors in E^{m+1}. We let A be a fixed unit vector in E^{m+1}, we denote the hyperplane passing through the origin in E^{m+1} which is perpendicular to A by E^m and we set $S^{m-1}=E^m \cap S^m$. We set $R=S^m \sim \{A\}$ and denote the stereographic projection from R to E^m by $L: R \to E^m$.

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1. Stereographic projection. The stereographic projection $L:R\to E^m$ is defined by

$$(1)\quad L(y) = A + h(y)(y - A), \quad y \in R, \text{ where } h(y) = 1/(1 - (y, A)).$$

One notices immediately that

$$(2)\qquad |y - A|^2 = 2/h(y) \quad\text{and}\quad h(y) = \tfrac{1}{2}(1 + |L(y)|^2).$$

Now let us consider the differential geometry of the map L. Let $e_1, \cdots, e_m$ be an orthonormal frame field on R with dual coframe field $w_1, \cdots, w_m$. Set $v_a = (A, e_a)$. (In this paper we use the index convention $1 \leqq a, b, c \leqq m$.) The following formulas are immediate.

$$(3)\qquad dy = \sum_a w_a e_a \quad\text{and}\quad dh = \sum_a h^2 v_a w_a,$$

$$(4)\qquad dL = \sum_a h^2 v_a w_a (y - A) + \sum_b h w_b e_b \qquad \text{(by (2) and (3))},$$

$$(5)\quad (dL, dL) = \sum_a h^2 w_a^2 \qquad \text{(by (2) and (4))}.$$

Thus, equation (5) tells us that L is a conformal equivalence between the Riemannian manifolds R and E^m with conformality factor h. In particular the forms $\theta_a = hw_a$ constitute an orthonormal coframe field for the flat Riemannian metric on R induced by the immersion L. Let w_{ab} be the connection forms on R corresponding to the frame field w_a defined by the structure equations [1, p. 62]

$$(6)\qquad dw_a = \sum_b w_{ab} \wedge w_b, \qquad w_{ab} = - w_{ba}.$$

Then since $d\theta_a = dh \wedge w_a + hdw_a = \sum_b (w_{ab} - v_b\theta_a + v_a\theta_b) \wedge \theta_b$ one sees that the induced flat connection on R has connection forms

$$(7)\qquad \theta_{ab} = w_{ab} - v_b\theta_a + v_a\theta_b.$$

Now suppose that M is an n-dimensional differentiable manifold, $2 \leqq n < m$, and that $X: M \to R$ is an immersion. Let $Y = L \circ X: M \to E^m$ be the composite immersion. First of all, let us perform some local calculations on the immersions (M, X) and (M, Y). In this local situation we can assume that $M \subset R$ and that X is the inclusion map. Also, we can assume that the frame field $e_1, \cdots, e_m$ is adapted along M, i.e., $e_1, \cdots, e_n$ are tangent to M at points of M and $e_{n+1}, \cdots, e_m$ are normal to M. Similarly if we set $f_a = dL(e_a)/h$ then the f_a form an orthonormal frame in E^m which is adapted along $Y(M)$.

Let us use the further index conventions $1 \leqq i, j, k \leqq n$, $n+1 \leqq r, s \leqq m$. If we restrict the adapted forms $w_a, \theta_a, w_{ab}, \theta_{ab}$ to the tangent spaces of M we see that $w_r = \theta_r = 0$. Thus, by the structure equations (6), applied to both w_r and θ_r, one sees that $0 = dw_r = \sum_a w_a \wedge w_{ar} = \sum_j w_j \wedge w_{jr}$ and $0 = d\theta_r = \sum_a \theta_a \wedge \theta_{ar} = \sum_j \theta_j \wedge \theta_{jr}$, so that by the Cartan Lemma [**3**, p. 18] we can write $w_{jr} = \sum_i b_{rij} w_i$ and $\theta_{jr} = \sum_i \beta_{rij} \theta_i$, where $b_{rij} = b_{rji}$ and $\beta_{rij} = \beta_{rji}$ are the matrices of the second fundamental forms of the immersions (M, X) and (M, Y), respectively. By (7) this implies

$$\beta_{rij} = (1/h) \circ b_{rij} - v_r \delta_{ij}. \tag{8}$$

Now by using (2) and (4) one sees that $v_r = (f_r, Y)/h$. Also, recalling that the mean curvature vector $\overline{H}$ and H of the immersions (M, X), (M, Y) respectively are defined by

$$(\overline{H}, e_r) = \frac{1}{n} \sum_j b_{rjj}, \qquad (H, f_r) = \frac{1}{n} \sum_j \beta_{rjj},$$

one sees that (8) and (2) imply

$$\begin{aligned} (H, f_r) &= (1/h)((\overline{H}, e_r) - (Y, f_r)) \\ &= 2((\overline{H}, e_r) - (Y, f_r))/(1 + |Y|^2). \end{aligned} \tag{9}$$

2. **An application to minimal submanifolds of spheres.** Let $X: M \to R$ and $Y = L \circ X$ be as above. We denote by dV the volume element induced on M by the immersion Y.

THEOREM 1. *Suppose that M is compact, connected and oriented and that $X: M \to R$ is a minimal immersion, i.e., $\overline{H} = 0$ on M. Then $\int_M (|Y|^2 - 1) dV \geqq 0$, with equality if and only if X immerses M into the hypersphere S^{m-1} of S^m perpendicular to A.*

COROLLARY. *If $m = n+1$ then $\int_M (|Y|^2 - 1) dV = 0$ if and only if X embeds M onto the hypersphere S^n; for by the theorem, X immerse into S^{m-1} and since $m-1 = n$ and M is compact and connected and S^n is simply-connected $(n \geqq 2)$ the corollary follows.*

The proof of the thorem rests on a computational lemma whose proof we leave as a simple exercise.

LEMMA. *Let Y be an immersion of the n-manifold M into E^m, $m > n$, such that $0 \notin Y(M)$. Let f be any twice continuously differentiable function defined on the positive real axis. Set $z = f(|Y|)$. Then*

$$\begin{aligned} \Delta z = ((&|Y| f''(|Y|) - f'(|Y|)) |Y^T|^2 \\ &+ n |Y|^2 f'(|Y|)(1 + (H, Y)))/|Y|^3 \end{aligned} \tag{*}$$

where Y^T is the tangential component of Y, H is the mean curvature vector and Δ is the Laplace-Beltrami operator induced on M by the immersion Y.

PROOF OF LEMMA. Just compute Δz.

COROLLARY (MINKOWSKI'S FORMULA). *If M is also compact and oriented then*

$$\int_M (1 + (H, Y))dV = 0.$$

PROOF. Set $f(t) = \frac{1}{2}t^2$ in the lemma and apply Stokes' theorem which says that $\int_M \Delta z dV = 0$.

PROOF of THEOREM. Set $f(t) = \frac{1}{4}t^4$ in the lemma. One computes that $\Delta z = 2|Y^T|^2 + n|Y|^2 + n|Y|^2(H, Y)$. By equation (9) and the hypothesis that (M, X) is minimal we see that $H = -2Y^N/(1+|Y|^2)$, that is, $H|Y|^2 = -2Y^N - H$, where Y^N is the component of Y normal to $Y(M)$. Substituting this fact into (*) and using Stokes' theorem we see that

$$\int_M (2|Y^T|^2 + n|Y|^2 - 2n|Y^N|^2 - n(H, Y))dV = 0.$$

Now by Minkowski's formula, $\int_M -n(H, Y)dV = \int_M n dV$, so the above equation becomes

$$\begin{aligned} 0 &= \int_M (2|Y^T|^2 + n|Y|^2 - 2n|Y^N|^2 + n)dV \\ &= 2(n+1)\int_M |Y^T|^2 dV + n\int_M (1 - |Y|^2)dV, \end{aligned}$$

since $|Y|^2 = |Y^T|^2 + |Y^N|^2$. Thus,

$$\int_M (|Y|^2 - 1)dV = 2\left(\frac{n+1}{n}\right)\int_M |Y^T|^2 dV \geqq 0.$$

Also, $\int_M (|Y|^2 - 1)dV = 0$ is equivalent to $Y^T \equiv 0$, which means that $Y(M)$ is contained in sphere centered at the origin, and thus $|Y| \equiv 1$. Under stereographic projection the unit sphere in E^m corresponds to the hypersphere perpendicular to A in S^m, so $X(M)$ lies in this hypersphere.

3. **Remarks on a result of Nomizu and Smyth.** The result referred to is in [2]. We state it as Proposition A.

PROPOSITION A. *Suppose that* (M, X) *is an immersion of a compact oriented hypersurface in* S^{n+1}, *the image of whose gauss map lies in a small hypersphere of* S^{n+1}. *Then* $X(M)$ *is a hypersphere* (*possibly a great hypersphere*) *of* S^{n+1}.

In this section we show that the Nomizu-Smyth result is equivalent to the following interesting result in Euclidean space.

PROPOSITION B. *Suppose that* (M, Y) *is an immersion of a compact oriented hypersurface into* E^{n+1} *with unit normal vector field* N *such that for some constant* $C>0$ *the map* $Y-C\cdot N$ *sends* M *into a sphere in* E^{n+1} *of radius* D *with* $D<C$. *Then* $Y(M)$ *is a hypersphere.*

REMARKS. If the map $Y-C\cdot N$ is nonsingular on M then this is already a classical result about parallel hypersurfaces which is true even locally. Also, any torus of revolution is a counterexample if we drop the hypothesis $D<C$.

THEOREM 2. *These propositions are equivalent.*

Proposition A→Proposition B. Suppose that (Y, M) is as in Proposition B. Then we may assume that the center of the sphere on which all the points $Y-CN$ lie is the origin. Thus, $|Y-CN|^2=D^2$ so that $|Y|^2-2CP+C^2=D^2$ or $2CP=C^2-D^2+|Y|^2$ where $P=(Y, N)$ is the support function for Y. If we set $\overline{Y}=Y/(C^2-D^2)^{1/2}$, then one gets $2\overline{P}/(1+|\overline{Y}|^2)=(C^2-D^2)^{1/2}/C$. One sees by equations (8) and 1.9 that the immersion $(M, \overline{Y})$ corresponds to an immersion (M, X) of M into the $(n+1)$-sphere whose Gauss map lies in a small hypersphere. Thus Proposition A applies and shows that (M, X) and, thus $(M, \overline{Y})$ and (M, Y), map M onto hyperspheres.

Proposition B→Proposition A. Suppose that (M, X) is as in Proposition A. Then there exists an element A of S^{n+1} and a constant $c>0$ such that $(\overline{N}, A)=c$, where $\overline{N}$ is the unit normal to X in S^{n+1}. Thus by equations (8) and (9) if Y is the corresponding immersion into R^{n+1} via stereographic projection then $c=2P/(1+|Y|^2)$. Thus $|Y-(1/c)N|^2=1/c^2-1$, and Proposition B now implies that (M, Y), and thus (M, X), map M onto a hypersphere.

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