SUBDIRECT SUMS, HEREDITARY RADICALS, AND STRUCTURE SPACES

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ABSTRACT. If a ring K is subdirectly embedded into the product S of a finite number of rings by a mapping i, then it is proved that $i(H(K))=i(K)\cap H(S)$ for any hereditary radical H, and that any structure space of K has the topology of a quotient space of a structure space of S.

1. **In'roduction.** In this note we establish two results which deal with finite subdirect sums of rings. (Rings, here, are associative but need not have a unity.) The first is concerned with hereditary radicals, the second with structure spaces.

Suppose that

$$K \xrightarrow{i} \prod_{j=1}^{n} K_j = S$$

is a subdirect embedding. That is, suppose that i is a ring monomorphism such that the composition of i with each projection p_t maps K *onto* K_t. We shall prove:

(1) For any hereditary radical property H, $i(H(K))=i(K)\cap H(S)$.

(2) Any structure space of K has the topology of a quotient space of a structure space of S.

Throughout this note, K, K_j $(j=1, 2, \cdots, n)$, i, and S will retain these meanings.

2. **Hereditary radicals.** For the definitions and properties of hereditary radicals, we refer to [1]. Our aim in this section is to prove

THEOREM 2.1. *If*

$$i: K \to S = \prod_{j=1}^{n} K_j$$

is a subdirect embedding, and if H is any hereditary radical, then $H(i(K))=i(K)\cap H(S)$.

The proof is broken into two steps. Throughout, we shall identify K_j with its natural image in S. The projection of S onto K_j will be

Received by the editors June 23, 1969.

AMS Subject Classifications. Primary 1630, 1650.

Key Words and Phrases. Subdirect sum, hereditary radical, structure space, Zariski topology.

denoted by $p_j, j=1, 2, \cdots, n$. Thus, for $x \in S$ we have $x=\sum_{j=1}^n p_j(x)$.

LEMMA 2.2. *For any hereditary radical H, $i(H(K)) \subset i(K) \cap H(S)$.*

PROOF. For each j, the composite map $p_j i$ maps K onto K_j. Therefore $p_j i(H(K)) \subset H(K_j)$. Thus, for $y \in H(K)$, $i(y)=\sum_{j=1}^n p_j i(y) \in \sum_{j=1}^n H(K_j)=H(S)$. ■

We know that $i(K) \cap H(S)$ is an ideal of $i(K)$. Once we also know that $i(K) \cap H(S)$ is an H-radical ring, it will follow that $i(K) \cap H(S) \subset H(i(K))=i(H(K))$, and the proof of Theorem 2.1 will be complete.

For each j, $p_j(i(K) \cap H(S))$ is an ideal of $p_j i(K)=K_j$, and this ideal is contained in $p_j(H(S))$ $(=H(K_j))$. Since H is hereditary, $p_j(i(K) \cap H(S))$ is itself an H-radical ring. It is easy to see that $i(K) \cap H(S)$ is a subdirect sum of the rings $p_j(i(K) \cap H(S))$, $j=1, 2, \cdots, n$. The next lemma shows that $i(K) \cap H(S)$ is indeed H-radical, and so the proof of Theorem 2.1 is complete.

LEMMA 2.3. *Let a ring P be a subdirect sum of H-radical rings $P_1, P_2, \cdots, P_n$, where H is a hereditary radical. Then P is also an H-radical ring.*

PROOF. We do the proof for the case $n=2$. The rest follows by induction. The assumptions imply that there are ideals T_1, T_2 of P such that $T_1 \cap T_2=0$ and $P_i \cong P/T_i$, $i=1, 2$.

Now $T_1 \cap T_2=0$, so $T_1 \cong (T_1+T_2)/T_2$, and the latter is H-radical, since it is an ideal of the H-radical ring P/T_2. Thus both T_1 and P/T_1 are H-radical rings. Since extensions of radical rings by radical rings are always radical, P is thus seen to be H-radical. ■

3. Structure spaces. Let P be a property of rings which is preserved under isomorphisms. For any ring K, define $F_P(K)$ to be $\{I : I$ is a prime ideal of K, $I \neq K$, and K/I has property $P\}$. The set $F_P(K)$ has a topology defined by the closure operation

$$\mathrm{cl}(U) = \{I \in F_p(K) : I \supseteq \sqcap U\}$$

for each subset U of $F_p(K)$. (Here the notation $\sqcap U$ means $\bigcap_{J \in U} J$.)

Various properties of these topological spaces, and the relations between the topological properties of the space and algebraic properties of the ring have been considered by several authors. (See, for example, [3].)

Our aim here is to establish the following result:

THEOREM 3.1. *Let*

$$K \xrightarrow{i} S = \prod_{j=1}^{n} K_j$$

be a subdirect embedding. Then the map

$$F_p(S) \xrightarrow{f} F_p(i(K))$$

defined by $f(I) = I \cap i(K)$, *is a continuous closed mapping onto* $F_p(i(K))$.

From this theorem, the following corollary is immediate.

COROLLARY 3.2. *Under the conditions of the theorem,* $F_P(K)$ *has the topology of a quotient space of* $F_P(S)$.

PROOF (OF THE COROLLARY). Clearly $F_P(K)$ and $F_P(i(K))$ are homeomorphic, since i is a ring monomorphism. The corollary follows at once from the theorem and [2, Theorem 3.8, p. 95]. ■

Theorem 3.1 will be proved in several steps.

LEMMA 3.3. *The mapping* f, *where* $f(I) = I \cap i(K)$, *is indeed a map of* $F_P(S)$ *into* $F_P(i(K))$.

PROOF. If I is any proper prime ideal of S, then, for some t, $K_t \not\subset I$. Since I is prime, $K_j \subset I$ for all $j \neq t$. It follows that $I = (I \cap K_t) + \sum_{j \neq t} K_j$. Therefore $S = I + K_t$.

The map $p_t i: K \to K_t$ is onto K_t, so for any $x \in S$ there is a k in K such that $p_t i(k) = p_t(x)$. Then $x - i(k) = \sum_{j \neq t} (p_j(x) - p_j i(k)) \in I$, so $S = i(K) + I$. Therefore $S/I \cong (i(K) + I)/I \cong i(K)/(i(K) \cap I)$. We see that for $I \in F_P(S)$, $f(I) = I \cap i(K) \in F_P(i(K))$. ■

LEMMA 3.4. *The mapping* f *is a continuous mapping onto* $F_P(i(K))$.

PROOF. The proof of the continuity of f is a standard result. To show that f is onto, for $t = 1, 2, \cdots, n$, let D_t be the ideal of K such that $i(D_t) = i(K) \cap (\sum_{j \neq t} K_j)$. This is the kernel of the mapping $p_t i$. Also, $\bigcap_{j=1}^n D_j = 0$.

Let $Q \in F_P(K)$. Then $Q \supset \prod_{j=1}^n D_j = 0$, so $Q \supset D_t$ for some t. Define Q' to be $p_t i(Q) + \sum_{j \neq t} K_j$. We shall show that $Q' \in F_P(S)$ and that $f(Q') = i(Q)$.

Now Q' is an ideal of S, and $S/Q' \cong K_t / p_t i(Q) = p_t i(K)/p_t i(Q) \cong K/Q$, where this last isomorphism follows since $Q \supset D_t = \ker(p_t i)$. This shows that Q' is indeed a member of $F_P(S)$.

Clearly we have $i(Q) \subset Q' \cap i(K)$. Conversely, let $x = i(k) \in Q' \cap i(K)$. Then there is an element q of Q for which $p_t i(q) = p_t(x) = p_t i(k)$. Then $i(q-k) = \sum_{j \neq t} p_t i(q-k) \in i(K) \cap (\sum_{j \neq t} K_j) = i(D_t) \subseteq i(Q)$. Therefore $i(k) \in i(Q)$, and we have $Q' \cap i(K) = i(Q)$. That is, $f(Q') = i(Q)$. ■

LEMMA 3.5. *The mapping* f *is a closed mapping.*

PROOF. As was seen in the proof of Lemma 3.3, for each proper prime ideal I of S, there is a t such that $I=(I\cap K_t)+\sum_{j\neq t} K_j$. Furthermore this t is easily seen to be unique. If we define $F_t=\{I\in F_P(S): I\supset \sum_{j\neq t} K_j\}$, for $t=1, 2, \cdots, n$, then $F_P(S)$ is the disjoint union of $F_1, F_2, \cdots, F_n$. Furthermore it is easily verified that each F_j is a closed subset of $F_P(S)$. If C is any closed subset of $F_P(S)$, then $f(C)=\bigcup_{j=1}^n f(F_j\cap C)$. To prove that f is closed, it suffices to show that, for a closed C, $f(F_j\cap C)$ is closed.

Let C be closed in $F_P(S)$, and let $i(Q)\in \mathrm{cl}(f(F_t\cap C))$. Then

$$i(Q)\supset \bigcap_{Q''\in F_t\cap C} [Q''\cap i(K)]\supset \left(\sum_{j\neq t} K_j\right)\cap i(K) = i(D_t),$$

where the D_j's are as in the proof of the previous lemma. From the proof of the previous lemma, we see that $Q'=p_t i(Q)+\sum_{j\neq t} K_j$ satisfies $f(Q')=i(Q)$. We will show that $Q'\in F_t\cap C$. Let $x=\sum_{j=1}^n p_t(x)$ be in $\sqcap(F_t\cap C)$. Since $p_t i$ is onto, there is a k in K such that $p_t i(k)=p_t(x)$. Since any $I\in F_t\cap C$ contains $\sum_{j\neq t} K_j$, it follows that $p_t i(k)=p_t(x)$ is in $\sqcap(F_t\cap C)$, and also that this intersection contains $i(k)$. Thus, for $I\in F_t\cap C$, we have $i(k)\in I\cap i(K)=f(I)$. Hence $i(k)\in \sqcap\{f(I): I\in F_t\cap C\}$. From the fact that $i(Q)\in \mathrm{cl}(f(F_t\cap C))$, we obtain $i(k)\in i(Q)$ and so $k\in Q$. Then $p_t(x)=p_t i(k)\in p_t i(Q)$, and so $x\in Q'$. This proves that $Q'\supset \sqcap(F_t\cap C)$, and so $Q'\in \mathrm{cl}(F_t\cap C)=F_t\cap C$. Finally, we have $i(Q)=f(Q')\in f(F_t\cap C)$, and so $f(F_t\cap C)$ is indeed closed. ■

This completes the proof of Theorem 3.1. We conclude by giving a simple example which shows that the mapping f need not be an open mapping.

For any ring K of characteristic p (p a prime), denote by $K^\#$ the ring formed by the Cartesian product $K\times Z_p$ (Z_p = integers mod p), where addition is componentwise and multiplication is defined by $(k, n)(k', n')=(kk'+nk'+n'k, nn')$.

Let S be a simple nontrivial Jacobson radical ring of characteristic p (see [4]), and consider the mapping

$$i: (S\oplus S)^\# \to S^\# \oplus S^\#,$$

where $i((s\oplus s', n))=(s, n)+(s', n)$. This is easily seen to be a subdirect embedding. If we choose P to be the property of being a prime ring, the mapping f of Theorem 3.1 induces a continuous closed map from $F_P(S^\#\oplus S^\#)$ onto $F_P((S\oplus S)^\#)$. It is not an open mapping, however, for the set $T=\{S^\#\oplus 0, S^\#\oplus S\}$ is a closed and open set in $F_P(S^\#\oplus S^\#)$, and $f(T)=\{S\oplus 0, S\oplus S\}$ is a closed but not open set in $F_P((S\oplus S)^\#)$.

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