AUTOMORPHISMS OF COUNTABLE PRIMARY ABELIAN GROUPS[1]

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ABSTRACT. It is proved that the automorphism group A of a countable primary abelian group G is transitive on certain subsets of subgroups of G. One such subset of subgroups in case G is without elements of infinite height is the collection of all basic subgroups of G with a fixed corank, finite or infinite.

In 1928 Shoda [7] gave a fairly complete description of the automorphism group of a finite abelian group. Freedman studied the structure of the automorphism group of a countable primary abelian group in [1]; there is, of course, a quick reduction from torsion groups to primary groups. Unlike Freedman, we do not attempt here to describe the whole structure of the automorphism group, but have as our aim a much smaller goal concerning the analysis of automorphisms. We prove that the automorphism group A of a countable primary abelian group G is transitive on a certain subset of distinguished subgroups of G. Surprisingly enough, the basic techniques are very similar to those of Freedman.

If H and K are arbitrary subgroups of G, we say that they are *equivalent* subgroups of G if there is an automorphism of G that maps H onto K. For the enumeration of the nonequivalent subgroups of small abelian groups see [3].

Kulikov proved that every primary abelian group G has a basic subgroup and that any two basic subgroups of G are isomorphic; see, for example, [2]. However, two basic subgroups of a given group G do not have to be equivalent subgroups of G. Even though the subgroups are isomorphic, they may be positioned in G quite differently. Indeed, the corresponding quotient groups are often not isomorphic. Let G be an unbounded, reduced, countable primary group and let B_1 and B_2 be basic subgroups of G such that $G/B_1 \cong G/B_2 \neq 0$. It follows from a theorem of Leptin [5] that not every isomorphism from B_1 onto B_2 can be extended to an automorphism of G, but we merely want *one* to extend. Tarwater [8] has proved, for the case that G is without elements of infinite height, that if G/B_1 has finite

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rank and $G/B_1 \cong G/B_2$ then B_1 and B_2 are equivalent subgroups of G. The following theorem puts Tarwater's result in a more general perspective.

THEOREM 1. *Let G be a countable primary abelian group and let B be a basic subgroup of G. Then B is uniquely determined, up to equivalence, as a subgroup of G by the numerical invariant*

$$r = \operatorname{rank}(G[p]/(B[p] + p^\omega G[p])).$$

We can still prove a more general, if less dramatic, result. Terminology and notation are standard; we refer to [2] and [4].

THEOREM 2. *Let G be a countable primary abelian group and let λ be an ordinal. Suppose that H and K are neat subgroups of G of length not exceeding λ such that $G[p] = \{H[p], p^{\alpha+1}G[p]\} = \{K[p], p^{\alpha+1}G[p]\}$ for each $\alpha < \lambda$. Then there exists an automorphism of G that (leaves $p^\lambda G$ fixed and) maps H onto K if and only if $H \cong K$ and*

$$G[p]/(H[p] + p^\lambda G[p]) \cong G[p]/(K[p] + p^\lambda G[p]).$$

PROOF. Since $p^\lambda G$ and $G[p]$ are always invariant subgroups of G, the conditions $H \cong K$ and

$$G[p]/(H[p] + p^\lambda G[p]) \cong G[p]/(K[p] + p^\lambda G[p])$$

are obviously necessary in order for an automorphism of G to map H onto K. Conversely, suppose that the conditions hold. We shall produce an automorphism of G that maps H onto K; it will be built up in stages. At the outset, we mention that $p^\alpha G \cap H = p^\alpha H$ and $p^\alpha G \cap K = p^\alpha K$ for each $\alpha \leqq \lambda$ according to [6, Lemma 1]. Therefore, an element of H or K has the same height in H or K as it does in G since the length of H and K does not exceed λ.

Suppose that $\pi: A \rightarrowtail\!\!\!\!\twoheadrightarrow B$ is an isomorphism from the subgroup A of G onto the subgroup B such that

(1) π preserves heights (computed in G),

and

(2) $\pi(A \cap H) = B \cap K$.

For the proper choice of A and B, we plan to show that π can be extended in such a way that conditions (1) and (2) continue to hold if π continues to denote the extension and A and B continue to denote the domain and image of the extended map.

We now define the initial choices of A and B. Their definitions hinge on whether $G[p]/(H[p]+p^\lambda G[p]) \cong G[p]/(K[p]+p^\lambda G[p])$ has finite or infinite rank. In case the rank is infinite, define $A = p^\lambda G = B$ and define $\pi: A \rightarrowtail\!\!\twoheadrightarrow B$ to be the identity map. In case the rank is finite, let

$$G[p] = S + H[p] + p^\lambda G[p] \quad \text{and} \quad G[p] = T + K[p] + p^\lambda G[p].$$

Define $A = S + p^\lambda G$ and $B = T + p^\lambda G$. We want an isomorphism π: $A \rightarrowtail\!\!\twoheadrightarrow B$ that extends the identity on $p^\lambda G$ and preserves heights (computed in G). By hypothesis, S and T are isomorphic, but we need a height-preserving isomorphism from S onto T. It is clear that such does not exist for arbitrary choices of S and T. However, we claim that S and T can be chosen such that there is a height-preserving isomorphism between them. First, consider the case that $\lambda - 1$ exists. Then

$$G[p] = H[p] + p^\lambda G[p] = K[p] + p^\lambda G[p]$$

and $S = 0 = T$, so there is no problem. Thus we may assume that λ is a limit ordinal. We distinguish two cases. First, suppose that H and K have length $\mu < \lambda$. Under the hypothesis of the theorem, it follows that S and T can be chosen such that every nonzero element has height in G less than λ but greater than μ. With S and T chosen this way, clearly there is a height-preserving isomorphism between them because

$$p^\mu G[p] = S + p^\lambda G[p] = T + p^\lambda G[p].$$

The case where the length of H and K is λ is only slightly more complicated. Let $\operatorname{rank}(S) = n = \operatorname{rank}(T)$. Choose a subgroup $H_0 = \sum_{i=1}^n \{x_i\}$ of $H[p]$ such that the height in H of x_i is α_i for $i \leqq n$ and such that $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \lambda$; this is possible for any n since H has length λ and λ is a limit ordinal. Since $H \cong K$, there exists a subgroup $K_0 = \sum_{i=1}^n \{y_i\}$ of $K[p]$ such that the height in K of y_i is also α_i. As we mentioned in the beginning of the proof, the height of an element in H is the same whether computed in H or G, and the same is true for K. Hence x_i and y_i have height exactly α_i in G. Choose $\beta < \lambda$ such that $\beta > \lambda_i$ for each $i \leqq n$. We can write, for any choice of S and T, $S = \sum_{i=1}^n \{s_i\}$ and $T = \sum_{i=1}^n \{t_i\}$ and we know that we can choose S and T contained in $p^\beta G$. Letting $s_i + x_i$ and $t_i + y_i$ replace s_i and t_i, we have the desired choice of S and T, $S = \sum_{i=1}^n \{s_i + t_i\}$ and $T = \sum_{i=1}^n \{t_i + y_i\}$. There is a height-preserving isomorphism between S and T now because $s_i + x_i$ and $t_i + y_i$ both have height α_i.

We have defined the initial choices of A and B and an isomorphism π: $A \rightarrowtail\!\!\twoheadrightarrow B$ that satisfies conditions (1) and (2); condition (2) is trivially satisfied because $A \cap H = 0 = B \cap K$. We now move to the induction step. Assume that the current A and B are finite extensions of the initial choices and that $\pi: A \rightarrowtail\!\!\!\twoheadrightarrow B$ is an isomorphism still satisfying conditions (1) and (2). In order to show that π can be extended further such that (1) and (2) continue to hold, six cases are distinguished. Common to all cases, are the following assumptions. We assume that x is an element outside of A but px is contained in A. Since A is a finite extension of $p^\lambda G$, there is an element $a \in A$ such that $x+a$ has maximum height in G among the elements of the coset $x+A$. Such an element is called proper with respect to A. There is no loss of generality in assuming that x itself is proper, and we shall do this. Let $h(x)$ denote the height of x; all heights from now on are computed with respect to G. Simplifying further, let $h(x)=\alpha$. Note that $\alpha<\lambda$. If $h(p(x+a))>\alpha+1$ for some $a \in A \cap p^\alpha G$, we can replace x by $x+a$ and still have a proper element. Thus we shall assume that $h(px)>\alpha+1$ if $h(p(x+a))>\alpha+1$ for some $a \in A \cap p^\alpha G$. With this assumption, there are two major cases.

Case 1. $h(px)=\alpha+1$.

Case 2. $h(px)>\alpha+1$.

Each of the above cases is divided into three subcases; subcase n of Case m is denoted by Case $m \cdot n$, $m \leqq 2$ and $n \leqq 3$. We mention in the beginning that each subcase of Case 1 is simplier to handle than the corresponding subcase of Case 2. We shall do the easy part first.

Case 1.1. $x \in H$. It follows that $px \in p^{\alpha+1}H = p^{\alpha+1}G \cap H$ and that $\pi(px) \in p^{\alpha+1}K = p^{\alpha+1}G \cap K$ in view of conditions (1) and (2). Thus $\pi(px)=py$ where $y \in p^\alpha K$. Extend π by mapping x onto y.

Case 1.2. $x+a \in H$ for some $a \in A$, $x \notin H$. Choose $y_0 \in p^\alpha G$ such that $py_0=\pi(px)$. If $y_0+\pi(a) \in K$, let $y=y_0$. If $y_0+\pi(a) \notin K$, consider $p(y_0+\pi(a))=\pi(p(x+a)) \in pK$. Letting $p(y_0+\pi(a))=pk_0$ with $k_0 \in K$, we have

$$y_0+\pi(a) = k_0+t, \quad \text{where } pt=0.$$

We can write $t=k_1+z$ where $k_1 \in K[p]$ and $z \in p^{\alpha+1}G$. Set $y=y_0-z$. Then $y+\pi(a) \in K$. Extend π by mapping x onto y.

Case 1.3. $x+a \notin H$ for all $a \in A$. Choose $y_0 \in p^\alpha G$ such that $py_0=\pi(px)$. If $y_0+b \notin K$ for all $b \in B$, let $y=y_0$. If $y_0+b \in K$ for some $b \in B$, let $b=\pi(a)$ and consider $p(x+a)$. π maps $p(x+a)$ onto $p(y_0+b) \in pK$. Therefore, $p(x+a) \in pG \cap H = pH$. Write

$$x+a = h_0+s, \quad \text{where } h_0 \in H \text{ and } ps=0.$$

Now $s \notin \{A, H\}$ (for Case 1.3), so in the decomposition $G[p]=S+H[p]+p^\lambda G[p]$ it follows that S is infinite. Otherwise, $A \supseteq S+p^\lambda G$, which is impossible. We know therefore that T is infinite where

$G[p]=T+K[p]+p^\lambda G[p]$ and that B is a finite extension of $p^\lambda G$. Hence there exists $t\in T$ such that $t+b\notin K$ for any $b\in B$. Choose $k\in K[p]$ such that $t+k\in p^{\alpha+1}G[p]$. Defining $y=y_0+t+k$, we have $y+b\notin K$ for all $b\in B$. Extend π by mapping x onto y.

Case 2.1. $x\in H$. In this case, $px\in p^{\alpha+2}G\cap H=p^{\alpha+2}H$. Let $px=ph$ where $h\in p^{\alpha+1}H$. Then $x-h\in G[p]$, has height exactly α, and is proper with respect to A. Since $\pi(A\cap H)=B\cap K$, since $A\cap H$ is finite, since $H\cong K$, and since π preserves heights not only in G but from H to K, we conclude that there must exist an element $z\in K[p]$ that is proper with respect to B and, like $x-h$, has height exactly α. Choose w in $p^{\alpha+1}K$ such that $pw=\pi(px)$ and set $y=w+z$. Extend π by mapping x onto y.

Case 2.2. $x+a\in H$ for some $a\in A$, $x\notin H$. Choose $w\in p^{\alpha+1}G$ such that $pw=\pi(px)$. There exists $z\in G[p]$ such that z is proper with respect to B and has height exactly α. Set $y_0=w+z$. If $y_0+\pi(a)\in K$, let $y=y_0$. If $y_0+\pi(a)\notin K$, we have to modify the definition of y. Observe that $p(y_0+\pi(a))=\pi(p(x+a))\in pK$, so $y_0+\pi(a)=k_0+t$ where $k_0\in K$ and $pt=0$. Moreover, we can write $t=k_1+v$ where $k_1\in K[p]$ and $v\in p^{\alpha+1}G[p]$. Define $y=y_0-v$ and note that $y+\pi(a)\in K$. Extend π by mapping x onto y.

Case 2.3. $x+a\notin H$ for all $a\in A$. As before, choose $w\in p^{\alpha+1}G$ such that $pw=\pi(px)$. Choose $z\in G[p]$ such that z is proper with respect to B and has height exactly α. Set $y_0=w+z$. If $y_0+b\notin K$ for all $b\in B$, let $y=y_0$. If $y_0+b\in K$ for some $b\in B$, again we need to modify the definition of y. Suppose $y_0+b\in K$ where $b\in B$ and let $\pi(a)=b$. Then $p(x+a)\in pH$ and

$$x+a=h_0+s, \quad \text{where } h_0\in H \text{ and } ps=0.$$

Now $s\notin\{A,H\}$ since $x\notin\{A,H\}$. By the same argument as that used in Case 1.3, there exists $t\in T$ such that $t\notin\{B,K\}$. Choose $k\in K[p]$ such that $t+k\in p^{\alpha+1}G[p]$. Define $y=y_0+t+k$ and observe that $y\notin\{B,K\}$. Extend π by mapping x onto y.

In each of the above cases, we claim that the extension of π is a height-preserving isomorphism from $\{A,x\}$ onto $\{B,y\}$ and that

$$\pi(\{A,x\}\cap H)=\{B,y\}\cap K;$$

details are left to the reader. Since G is countable and since H and K, as well as A and B, are symmetrical with respect to the hypothesis, π can be extended to an automorphism of G that maps H onto K. This completes the proof of Theorem 2.

Theorem 1 can be obtained at once from Theorem 2 with $\lambda=\omega$. Theorem 2 also encompasses the following.

THEOREM 3. *Let G be a countable torsion abelian group and let λ be an ordinal. If each of H and K is maximal in G with respect to having trivial intersection with $p^{\lambda}G$, then there is an automorphism of G that maps H onto K.*

COROLLARY. *All high subgroups of a countable torsion group are equivalent.*

The preceding corollary suggests the following problem.

PROBLEM. Characterize those groups whose high subgroups are all equivalent.

Finally, we remark that Theorem 2 can be interpreted as a partial solution to Problem 52 in [2].

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