# ON THE MEASURE OF ZERO SETS OF COORDINATE FUNCTIONS

JOHN E. COURY[1]

Let $G$ be a non-Abelian compact topological group, with normalized Haar measure $\mu_G$, and suppose that $V \neq 1$ is a continuous unitary irreducible representation (CUIR) of $G$ on a Hilbert space $H$ of dimension $d \geqq 2$. Let $\{\xi_1, \cdots, \xi_d\}$ be an orthonormal basis of $H$ and $v_{pq}(x) = \langle V_x\xi_q, \xi_p\rangle$ a coordinate function of $V$. This paper investigates the measure of the set $C = \{x \in G: v_{pq}(x) = 0\}$. A sufficient condition for $C$ to have measure zero for every choice of $V$, $\xi_p$, and $\xi_q$ is that $G$ be connected. An example is given to show that this condition is not necessary. If, however, $G$ is totally disconnected, then there always exist such representations of $G$, of dimension at least two, for which the zero set of every coordinate function has positive measure.

We require the following lemmas. For the proof of Lemma 1, see [1].

LEMMA 1. *Let $f(x)$ be analytic in the open set $I \subset R^n$ mapping into the Banach space $K$ over $R$. Suppose $f$ is not identically zero. Then $D = \{x \in I: f(x) = 0\}$ has $n$-dimensional Lebesgue measure* 0.

The next lemma is based on a discussion in [4, pp. 161–170], wherein the Haar integral for a connected Lie group $G$ is given in terms of an integral over the underlying manifold of $G$.

LEMMA 2. *Let $G$ be a connected Lie group, with underlying manifold of dimension $p$, and let $(V, \phi)$ be a local chart for a fixed (but arbitrary) $x$ in $G$. Let $D$ be a relatively closed subset of $\phi(V)$ with $p$-dimensional Lebesgue measure zero in $R^p$.*

*Then $\phi^{-1}(D)$ has Haar measure zero in $G$.*

PROOF. Denote $p$-dimensional Lebesgue measure by $\lambda^p$. Let $A \subset \phi(V)$ be compact, with $\lambda^p(A) = 0$. Then we can find a sequence $\{W_n\}_{n=1}^{\infty}$ of open sets in $\phi(V)$ such that

(i) $A \subset W_n \subset W_n^- \subset \phi(V)$, $W_n^-$ compact;

(ii) $W_{n+1} \subset W_{n+1}^- \subset W_n$;

(iii) $\lambda^p(W_n) < 1/n$.

Let $f_n: G \to [0, 1]$ be a continuous function which is 1 on $\phi^{-1}(W_n)$ and 0 outside of $V$. Denote Haar measure on $G$ by $\mu$ and the characteristic

Received by the editors May 9, 1969.

[1] The author expresses his thanks to the referee for his valuable comments and suggestions.

function of a set $B$ by $\xi_B$. Appealing to [4, pp. 161–170], we may write

$$\int_V f d\mu = \int_{\phi(V)} (f \circ \phi^{-1}) \cdot F d\lambda^p,$$

where $F$ is a continuous function on $\phi(V)$ and $f$ is a continuous function on $G$ vanishing outside $V$. $F$ is bounded on $W_1^-$, say by $M$, and so is bounded by $M$ on each $W_n^-$. Hence for each $n$, $(f_n \circ \phi^{-1}) \cdot F$ is bounded by $M$ on $W_n$. Thus we infer that

$$\begin{aligned} \mu(\phi^{-1}(A)) &= \int_G \xi_{\phi^{-1}(A)} d\mu = \int_{\phi^{-1}(W_n)} \xi_{\phi^{-1}(A)} d\mu \\ &\leqq \int_{\phi^{-1}(W_n)} f_n d\mu = \int_{W_n} (f_n \circ \phi^{-1}) \cdot F d\lambda^p \\ &\leqq M \cdot \lambda^p(W_n) < M/n \qquad \text{for every } n. \end{aligned}$$

Hence $\mu(\phi^{-1}(A)) = 0$.

Now choose a sequence $\{F_n\}_{n=1}^{\infty}$ of compact cubes in $\phi(V)$ such that $\phi(V) = \bigcup_{n=1}^{\infty} F_n$, and set $D_n = D \cap F_n$. Because $D$ is relatively closed in $\phi(V)$, $D_n$ is relatively closed in $F_n$ and so is compact. It follows from the first part of the proof that $\mu(\phi^{-1}(D_n)) = 0$ for each $n$. Since $D = \bigcup D_n$, we conclude that $\phi^{-1}(D)$ has Haar measure zero.

LEMMA 3. *Let $K$ be a compact connected Lie group and $g$ a nonzero analytic function on $K$. Then $B = \{y \in K : g(y) = 0\}$ has Haar measure zero.*

PROOF. Suppose that the interior $B^0$ of $B$ is nonvoid and let $x \in B^{0-}$. Choose a local chart $(U_x, \phi_x)$ for $x$, $\phi_x(U_x)$ being an open cube in $R^s$ and $\phi_x$ a homeomorphism. Since $g$ is analytic, so also is the function $g \circ \phi_x^{-1} : \phi_x(U_x) \to \mathbf{C}$, by definition (see [4, p. 74]). Plainly, $\phi_x(U_x)$ is connected and open; since $U_x \cap B^0 \neq \emptyset$, $\phi_x(U_x \cap B^0)$ is open. Moreover, since $g \circ \phi_x^{-1}$ vanishes on the open set $\phi_x(U_x \cap B^0) \subset \phi_x(U_x)$, it vanishes identically on $\phi_x(U_x)$ (see [5, p. 202]). Thus $g$ vanishes on $U_x$, whence $U_x \subset B^0$. Since $x$ was arbitrary, it follows that $B^{0-}$ is open. But then we have that $B^{0-} = K$ since $K$ is connected, and hence that $B = K$. This contradicts the fact that $g$ is nonzero. We conclude that $B^0 = \emptyset$.

For a fixed $y \in B$ and a local chart $(U_y, \phi_y)$ of $y$, set $W = \phi_y(U_y)$. Then $g \circ \phi_y^{-1}$ cannot be identically zero on $W$, by the preceding paragraph. The set $D_y = \{x \in W : g \circ \phi_y^{-1}(x) = 0\}$ therefore has $s$-dimensional Lebesgue measure zero, by Lemma 1. $D_y$ is closed in the relative topology of $W$ as a subspace of $R^s$.

Since $B$ is closed and hence compact, $B$ can be covered by a finite number of open sets $U_y$, with $y \in B$. Because $\phi_y$ is a homeomorphism, we have that $g(t)=0$ for $t \in U_y$ if and only if $g \circ \phi_y^{-1}(\phi_y(t))=0$; thus $U_y \cap B = \phi_y^{-1}(D_y)$. Since $\phi_y^{-1}(D_y)$ has Haar measure zero in $K$, by Lemma 2, we infer that $B$ has Haar measure zero.

THEOREM 1. *Let $G$ be a connected compact group and let $H$ be a normal subgroup such that $G/H$ is a Lie group. Suppose that $f$ is a nonzero complex-valued function on $G$ which assumes constant values on the left cosets of $H$ such that the induced mapping $\bar{f}: G/H \to \mathbf{C}$ is real analytic. Then the set $C = \{x \in G : f(x) = 0\}$ has Haar measure zero.*

PROOF. Since the factor group $G/H$ is connected and compact, we may apply Lemma 3 to conclude that the zero set $B$ of the function $\bar{f}$ has Haar measure zero in $G/H$. Appealing to [**3**, (28.54)], we may write

$$\int_{G/H} h d\mu_{G/H} = \int_G h \circ \pi d\mu_G \tag{1}$$

for $h \in C_{00}(G/H)$, where $\pi$ is the canonical mapping between $G$ and $G/H$. From this we easily infer that (1) obtains with $h = \xi_B$, the characteristic function of $B$. Since $f(x)=0$ if and only if $\bar{f}(xH)=0$, we may write the set $C$ as $\pi^{-1}(B)$. Then we have that $\xi_C = \xi_B \circ \pi$ and hence

$$0 = \mu_{G/H}(B) = \int_{G/H} \xi_B d\mu_{G/H} = \int_G \xi_B \circ \pi d\mu_G = \int_G \xi_C d\mu_G = \mu_G(C).$$

Thus $C$ has measure zero, concluding the proof.

We now prove that zero sets of coordinate functions have zero measure in compact connected groups.

THEOREM 2. *Let $G$, $V$, and $C$ be as in the opening paragraph, and suppose that $G$ is connected. Then $C$ has Haar measure zero for every choice of $\xi_p$ and $\xi_q$.*

PROOF. Since the representation space of $V$ is finite-dimensional [**2**, (22.13)], $V(G)$ may be viewed as a subgroup of the Lie group $U(d)$, where $U(d)$ denotes the group of $d \times d$ unitary matrices over the complex field $\mathbf{C}$. Let $H = \ker V = \{x \in G : V_x = I\}$; then $H$ is a normal subgroup of $G$, of measure zero. The factor group $G/H$ is connected and compact; the mapping $V^H: G/H \to V(G)$, defined by $V^H(xH) = V_x$, is clearly a topological isomorphism.

Being a compact and therefore closed subgroup of $U(d)$, $V(G)$ is itself a Lie group, whence $G/H$ is a Lie group (see [**3**, pp. 130, 135]);

thus $V^H$ is analytic [6, p. 84]. Define $v^H_{pq}: G/H \to \mathbf{C}$ by $v^H_{pq}(xH) = \langle V^H(xH)\xi_q, \xi_p\rangle = \langle V_x\xi_q, \xi_p\rangle$, for arbitrary elements $\xi_p$ and $\xi_q$ in an orthonormal basis of $R^s$, for some $s \leqq d^2$; thus $v^H_{pq}$ is the function on $G/H$ induced by $v_{pq}$. Write $v^H_{pq} = g \circ V^H$, where $g(V_x) = \langle V_x\xi_q, \xi_p\rangle$. The function $g$ is analytic, being the restriction to $V(G)$ of the analytic function $g'(U) = \langle U\xi_q, \xi_p\rangle$ on $U(d)$. Thus $v^H_{pq}$ is analytic.

We note that $v_{pq}(x)$ cannot be identically zero, in view of the fact that the square of the absolute value of any coordinate function integrates to a positive number. Theorem 1 now applies, with $f = v_{pq}$ and $\bar{f} = v^H_{pq}$, and we conclude that the set $C = \{x \in G: \langle V_x\xi_q, \xi_p\rangle = 0\}$ has Haar measure zero in $G$.

That the converse of the previous theorem does not obtain is demonstrated by the next example.

Example. Let $H$ be a compact disconnected Abelian group and $K$ a compact connected non-Abelian group. Consider the disconnected group $G = H \times K$. Continuous unitary irreducible representations $V$ of $G$ are of the form $\chi \cdot W$, where $\chi$ is a character of $H$ and $W$ is a representation on $K$. Thus the zero set of a coordinate function of $V$ is precisely the zero set of a coordinate function of $W$, hence of measure zero by Theorem 2.

A partial converse of the theorem, however, can be proved. The following lemma is required.

Lemma 4. *Let $G$ be a compact non-Abelian group, and suppose that $G$ admits a finite non-Abelian homomorphic image.*

*Then there exists a CUIR $V$ of $G$, of dimension at least 2, such that every coordinate function of $V$ has a zero set of positive measure.*

Proof. Write $K = f(G)$, where $f$ is a homomorphism and $K$ is a finite non-Abelian group. Set $N = \ker f$; then $G/N$ is isomorphic with $K$. By [2, (5.26)], $N$ is open and so has positive Haar measure. Since $G/N$ is non-Abelian, there exists a nontrivial CUIR $W$ of $G/N$ of dimension $d \geqq 2$. Let $H$ denote the representation space of $W$, and let $\xi_p$ and $\xi_q$ be arbitrary elements of an orthonormal basis of $H$, with $p \neq q$. Then evidently we have $\langle W_N\xi_q, \xi_p\rangle = 0$ since $W_N = I$.

Denote by $\pi$ the canonical homomorphism between $G$ and $G/N$. Then $V = W \circ \pi$ is a continuous unitary representation of $G$ and is irreducible since $W$ is. Further, for any $x \in N$, we have $\langle V_x\xi_q, \xi_p\rangle = \langle W_N\xi_q, \xi_p\rangle = 0$, whence we infer that $N \subset C = \{x \in G: v_{pq}(x) = 0\}$. Thus $C$ has positive measure.

Finally we consider the case $p = q$. Then for any $x \in C$, we clearly have $xN \subset C$ and so $C$ is of positive measure. Thus every coordinate function of $V$ has a zero set of positive Haar measure.

THEOREM 3. *Let $G$ be a compact non-Abelian group satisfying one of the following conditions*:

(i) *$G$ is finite*;

(ii) *$G$ is infinite and totally disconnected*;

(iii) *$G$ is infinite, disconnected, and the commutator subgroup $G_1$ of $G$ is dense in $G$.*

*Then there exists a CUIR $V$ of $G$, of dimension greater than one, such that the zero set of every coordinate function of $V$ has positive measure.*

PROOF. In view of Lemma 4, it suffices to show that each of the above conditions implies the existence of a finite non-Abelian homomorphic image of $G$. For $G$ finite this is obvious. Now suppose that $G$ is infinite and totally disconnected. Let $x, y \in G$ be such that $xy \neq yx$, and let $U$ be a neighborhood of the identity element $e$ which does not contain $xyx^{-1}y^{-1}$. Since $G$ is totally disconnected, there is an open normal subgroup $N$ of $G$ contained in $U$ [2, (7.7)], and so $xyx^{-1}y^{-1} \notin N$. Thus we have $xNyN \neq yNxN$ and hence $G/N$ is non-Abelian. Since $N$ is open, $G/N$ is finite.

Finally, suppose that $G$ is infinite, disconnected, and $G_1$ is dense in $G$. Denote the component of $e$ by $C(e)$; then $C(e)$ is a closed normal subgroup of $G$ and $G/C(e)$ is totally disconnected [2, (7.1) and (7.3)]. If $G/C(e)$ were Abelian, then $G_1 \subset C(e)$ and so $G = G_1^- \subset C(e)^- = C(e)$, a contradiction since $G$ is not connected. Thus $G/C(e)$ is non-Abelian and totally disconnected, and so, by the previous paragraph, $G/C(e)$ (and hence $G$) has a finite non-Abelian homomorphic image.

## REFERENCES


1. R. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965. MR **31** #4927.

2. E. Hewitt and K. Ross, *Abstract harmonic analysis*. Vol. I: *Structure of topological groups. Integration theory, group representations*, Die Grundlehren der math. Wissenschaften, Bd. 115, Academic Press, New York and Springer-Verlag, Berlin and New York, 1963. MR **28** #158.

3. ———, *Abstract harmonic analysis*. II, Springer-Verlag, Berlin, (to appear).

4. C. Chevalley, *Theory of Lie groups*. I, Princeton Math. Series, vol. 8, Princeton Univ. Press, Princeton, N. J., 1946. MR **7**, 412.

5. J. Dieudonné, *Foundations of modern analysis*, Pure and Appl. Math., vol. 10, Academic Press, New York, 1960. MR **22** #11074.

6. G. Hochschild, *The structure of Lie groups*, Holden-Day, San Francisco, Calif., 1965. MR **34** #7696.


UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98105 AND
UNIVERSITY OF MASSACHUSETTS, AMHERST, MASSACHUSETTS 01002