THE PERRON INTEGRAL AND EXISTENCE AND UNIQUENESS THEOREMS FOR A FIRST ORDER NONLINEAR DIFFERENTIAL EQUATION

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ABSTRACT. The Perron integral is used to establish an existence and uniqueness theorem concerning the initial value problem $y'(t)=f(t, y(t))$, and $y(t_0)=\alpha$, for t on the interval $I=\{t|0\leqq t\leqq 1\}$. The existence and uniqueness of the solution is obtained by use of a generalized Lipschitz condition, and a Picard sequence which is equiabsolutely continuous on I. Also, we prove a theorem on the uniqueness of solution by a generalization of Gronwall's inequality.

1. **Introduction.** This paper deals with existence and uniqueness theorems concerning the i.v.p. (initial value problem)

$$y'(t) = f(t, y(t)), \qquad y(t_0) = \alpha, \tag{1}$$

where $f(t, y(t))$, for any continuous y on $I=\{t|0\leqq t\leqq 1\}$, is defined a.e. (almost everywhere) on I. There is an extensive body of literature dealing with conditions under which solutions for (1) exist. In most discussions f is taken to be integrable in the Lebesgue sense. Here we use the Perron integral. It was shown by Bauer in [1] (see also Kamke [5], McShane [8], Saks [10]) that the Perron definition of the integral leads to a generalization of the Lebesgue integral. Northcutt [9] used the Perron integral, and obtained solutions for (1). The author [6] has also considered the Perron integral, and established an existence and uniqueness theorem for a second order nonlinear partial differential equation.

2. **Preliminary theorems.** Integration throughout this paper is in the Perron sense and the following theorems will be used.

THEOREM 2.1 (KAMKE [5, p. 210]). *If $f\subset P$ (Perron integrable) on I and $f(t)\geqq 0$ a.e. on I, then, $f\subset \mathfrak{L}$ (Lebesgue integrable) on I, and for t on I, $\int_0^t f=(\mathfrak{L})\int_0^t f$.*

COROLLARY 2.1.1. *If f and $g\subset P$ on I and $f(t)\geqq g(t)$ a.e. on I, then, for $0\leqq t_0<t\leqq 1$, $\int_{t_0}^t f\geqq \int_{t_0}^t g$.*

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COROLLARY 2.1.2. *If* $f \subset P$ *and* $g \subset \mathfrak{L}$ *on* I *and* $f(t) \geqq g(t)$ *a.e. on* I, *then* $f \subset \mathfrak{L}$ *on* I.

THEOREM 2.2. *If* $f \subset P$ *on* I *and* $g \subset$ B.V. (*bounded variation*) *on* I *then* $f \cdot g \subset P$ *on* I *and* $\int_0^t f \cdot g = F(t)\, g(t) - \int_0^t F dg(s)$ *where* $F(t) = \int_0^t f$.

See Gordon and Lasher [3] for an elementary proof.

3. Existence theorems. We prove the following

THEOREM 3.1. *If*

H1. $f(t, y)$ *is continuous in* y *for* t *a.e. on* I.

H2. $f(t, y(t)) \subset P$ *on* I *for* y *continuous on* I.

H3. $f(t, y(t)) \geqq g(t)$ *a.e. on* I *where* $g \subset P$ *on* I.

H4. $|f(t, y(t)) - f(t, y^*(t))| \leqq v(t)|y(t) - y^*(t)|$ *a.e. on* I *where* $v \subset P$ (*and hence in* $\mathfrak{L}$) *on* I. *Then there exists for the* i.v.p. (1), *a Picard sequence, which yields a solution,* $\psi(t)$, *which is continuous and locally absolutely continuous* (LAC) *on* I, *only if the sequence* $\{\int_0^t (f_n - g)\}$ *is* EAC (*equiabsolutely continuous*) *on* I.

LEMMA 3.1. *If*

H1. $f_n \subset P$ *on* I *for each counting number* n.

H2. $\lim_n f_n(t) = f(t)$ *a.e. on* I.

H3. $f_n(t) \geqq g(t)$ *a.e. on* I *for each* n *where* $g \subset P$ *on* I. *Then,* $f \subset P$ *on* I, *and* $\lim_n \int_0^t f_n = \int_0^t f$ *only if the sequence* $\{\int_0^t [f_n - g]\}$ *is* EAC *on* I.

The proof of this lemma (see [7]) is based on a corresponding theorem by Vitali [11] for functions integrable in the Lebesgue sense. Briefly, by Theorem 2.2, $(f_n - g) \subset \mathfrak{L}$ on I and from [11]

$$\lim_n (\mathfrak{L}) \int_0^t [f_n - g] = (\mathfrak{L}) \int_0^t [f - g].$$

Consequently, $(f - g) \subset P$, and $\lim_n \int_0^t [f_n - g] = \int_0^t [f - g]$ and since $g \subset P$ on I, then $f \subset P$ on I and $\lim_n \int_0^t f_n = \int_0^t f$.

PROOF OF THEOREM 3.1. Let $y_0(t) = \alpha$;

$$y_{n+1}(t) = \int_{t_0}^t f(s, y_n(s))ds + \alpha \qquad (n = 0, 1, 2, \cdots). \tag{2}$$

We note that $y_1(t)$ is continuous and LAC (see Saks [10, p. 251]) on I. By induction, we have that $y_n(t)$ is continuous and LAC on I for each counting number n. Define

$$u_n(t) = y_{n+1}(t) - y_n(t) \qquad (n = 0, 1, 2, \cdots). \tag{3}$$

Then, $u_0(t) = \int_{t_0}^t f(s, \alpha)ds$ and

$$(4) \qquad u_n(t) = \int_{t_0}^{t} [f(s, y_n(s)) - f(s, y_{n-1}(s))] ds.$$

Under H4, the integral in (4) may be taken in the Lebesgue sense. Now, since $u_0(t)$ is continuous on I, then there exists a number k such that for t on I. $|u_0(t)| < k$ and $|u_1(t)| < k\int_0^t v$ and in general, since $v(t) \geqq 0$ a.e. on I

$$(5) \qquad |u_n(t)| < k\left[\int_0^t v\right]^n / n! \leqq k\left[\int_0^1 v\right]^n / n!.$$

Consequently, $\sum_{i=0} |u_i(t)|$ converges uniformly on I. But, $\sum_{i=0}^{n} u_i(t) = y_{n+1}(t) - y_0(t)$. Hence, there exists a function $\psi(t)$ such that $\lim_n y_n(t) = \psi(t)$ uniformly on I where $\psi(t)$ is continuous and LAC on I. From H1 we have $\lim_n f(t, y_n(t)) = f(t, \psi(t))$ a.e. on I.

Then, H3, and Lemma 3.1, yield

$$\psi(t) = \int_{t_0}^{t} f(s, \psi(s)) ds + \alpha$$

where $\psi(t_0) = \alpha$ and $\psi'(t) = f(t, \psi(t))$ a.e. on I. Q.E.D.

Under the hypotheses H1–H3 of Theorem 3.1, and by use of the Cauchy-Euler method, Northcutt [9] showed that there exists a function $\psi(t)$, continuous and LAC, which is a solution of the i.v.p. (1) only if the sequence $\{\int_0^t [f_n - g]\}$ is EAC on I. His method of proof is based on Ascoli's theorem on a uniformly bounded set of equicontinuous functions on I, and on Lemma 3.1. Uniqueness is not to be expected in this case.

4. Uniqueness theorems.

THEOREM 4.1. *The solution $\psi(t)$ in Theorem 3.1 is unique.*

PROOF. Assume that there exists for the i.v.p. (1) another solution $\psi^*(t)$. Let $Y(t) = \psi(t) - \psi^*(t)$ for t on I. Then,

$$Y(t) = \int_{t_0}^{t} [f(s, \psi(s)) - f(s, \psi^*(s))] ds$$

and

$$0 \leqq |Y(t)| \leqq \lim_n k\left[(\mathcal{L}) \int_0^1 v\right]^n / n!.$$

Hence $Y(t) = 0$ for t on I and uniqueness of solution for (1) follows.

The following involve generalizations of some of the results of Ettlinger [2].

THEOREM 4.2. *If* $\psi_1(t)$ *and* $\psi_2(t)$ *are* LAC *on* I *and satisfy the* i.v.p. (1) *in a region* $R=\{(t,y)\mid 0\leqq t\leqq 1, \text{ all } y\}$. *If further*, $\psi_1(t)$ *and* $\psi_2(t)$ *satisfy* H2 *and* H4 *of Theorem* 3.1 *on* I. *Then*, $\psi_1(t)=\psi_2(t)$ *on* I.

A proof of this theorem may be obtained by use of the following lemma, which is a generalization of Gronwall's lemma [4].

LEMMA 4.2.1. *If* $h\subset\mathfrak{L}$ *and* $g\subset P$ *on* I, x *is* LAC *on* I, *and satisfy the differential inequality*

$$x'(t)+h(t)x(t)\leqq g(t), \text{ a.e. on } I. \tag{6}$$

Then, for $0\leqq t_0<t\leqq 1$,

$$x(t)\leqq \exp\left(-\int_{t_0}^{t}h\right)\left[\int_{t_0}^{t}g\exp\left(\int_{t_0}^{s}h\right)+x(t_0)\right]. \tag{7}$$

PROOF. Since $\exp(\int_{t_0}^{t}h)>0$ we have

$$x'(t)\exp\left(\int_{t_0}^{t}h\right)+h(t)x(t)\exp\left(\int_{t_0}^{t}h\right)\leqq g(t)\exp\left(\int_{t_0}^{t}h\right). \tag{8}$$

Now, $g\subset P$ and $\exp(\int_{t_0}^{t}h)$ is absolutely continuous on I. Then, by Theorem 2.2, $g\cdot\exp(\int_{t_0}^{t}h)\subset P$ on I. Furthermore,

$$\frac{d}{dt}\left[x(t)\exp\left(\int_{t_0}^{t}h\right)\right]=x'(t)\exp\left(\int_{t_0}^{t}h\right)+h(t)x(t)\exp\left(\int_{t_0}^{t}h\right) \text{ a.e. on } I.$$

Hence, by Corollary 2.1.1, and (8), we have for $t\geqq t_0$

$$\left[x(s)\exp\left(\int_{t_0}^{s}h\right)\right]_{t_0}^{t}\leqq\int_{t_0}^{t}g(s)\exp\left(\int_{t_0}^{s}h\right), \tag{9}$$

and statement (7) is thus obtained. We note that the equality in (7), $\phi(t)=\exp(-\int_{t_0}^{t}h)[\int_{t_0}^{t}g\exp(\int_{t_0}^{s}h)+\phi(t_0)]$, represents a solution for the linear differential equation $x'(t)+h(t)x(t)=g(t)$.

COROLLARY 4.2.1. *If* $x(t_0)=0$ *and, a.e. on* I, $g(t)=0$ *and* $x(t)\geqq 0$, *then* $x(t)=0$ *on* $[t_0,1]$.

PROOF OF THEOREM 4.2. Let $x(t)=|\psi_1(t)-\psi_2(t)|$ on I. Then, $x'(t)\leqq v(t)x(t)$ a.e. on I. Since $x(t_0)=0$, then for $t\leqq t_0$ and by Corollary 2.1.1

$$x(t)\geqq\int_{t_0}^{t}vx. \tag{10}$$

Now, from H4 and by Corollary 2.1.2, $|f(t, \psi_1(t)) - f(t, \psi_2(t))| \subset P$ on I, and is also integrable in the Lebesgue sense. Hence, for t on I, H4 yields

$$x(t) \leqq \int_{t_0}^{t} vx. \tag{11}$$

From (10) and (11) we have on $[0, t_0]$

$$x(t) = \int_{t_0}^{t} vx$$

and $x'(t) = v(t)x(t)$ a.e. on $[0, t_0]$. Corollary 4.2.1 then yields

$$x(t) = 0 \quad \text{on } I.$$

Hence, $\psi_1(t) = \psi_2(t)$ on I.

5. **Remark.** In equation (1), $y(t)$ may be considered as a real valued vector function defined in a Euclidean space of n dimensions. Conclusions follow as in the scalar case.

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