THE PERRON INTEGRAL AND EXISTENCE AND UNIQUENESS THEOREMS FOR A FIRST ORDER NONLINEAR DIFFERENTIAL EQUATION

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Abstract. The Perron integral is used to establish an existence and uniqueness theorem concerning the initial value problem $y'(t) = f(t, y(t))$, and $y(t_0) = \alpha$, for $t$ on the interval $I = \{t | 0 \leq t \leq 1\}$. The existence and uniqueness of the solution is obtained by use of a generalized Lipschitz condition, and a Picard sequence which is equiabsolutely continuous on $I$. Also, we prove a theorem on the uniqueness of solution by a generalization of Gronwall's inequality.

1. Introduction. This paper deals with existence and uniqueness theorems concerning the i.v.p. (initial value problem)

$$(1) \quad y'(t) = f(t, y(t)), \quad y(t_0) = \alpha,$$

where $f(t, y(t))$, for any continuous $y$ on $I = \{t | 0 \leq t \leq 1\}$, is defined a.e. (almost everywhere) on $I$. There is an extensive body of literature dealing with conditions under which solutions for (1) exist. In most discussions $f$ is taken to be integrable in the Lebesgue sense. Here we use the Perron integral. It was shown by Bauer in [1] (see also Kamke [5], McShane [8], Saks [10]) that the Perron definition of the integral leads to a generalization of the Lebesgue integral. Northcutt [9] used the Perron integral, and obtained solutions for (1). The author [6] has also considered the Perron integral, and established an existence and uniqueness theorem for a second order nonlinear partial differential equation.

2. Preliminary theorems. Integration throughout this paper is in the Perron sense and the following theorems will be used.

Theorem 2.1 (Kamke [5, p. 210]). If $f \in P$ (Perron integrable) on $I$ and $f(t) \geq 0$ a.e. on $I$, then, $f \in L$ (Lebesgue integrable) on $I$, and for $t$ on $I$, $\int_0^t f = (L)\int_0^t f$.

Corollary 2.1.1. If $f$ and $g \in P$ on $I$ and $f(t) \geq g(t)$ a.e. on $I$, then, for $0 \leq t_0 < t \leq 1$, $\int_{t_0}^t f \geq \int_{t_0}^t g$.  

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Corollary 2.1.2. If \( f \subseteq P \) and \( g \subseteq \mathcal{L} \) on \( I \) and \( f(t) \geq g(t) \) a.e. on \( I \), then \( f \subseteq \mathcal{L} \) on \( I \).

Theorem 2.2. If \( f \subseteq P \) on \( I \) and \( g \subseteq B.V. \) (bounded variation) on \( I \) then \( f \cdot g \subseteq P \) on \( I \) and \( \int_0^t f \cdot g = F(t) g(t) - \int_0^t F dg(s) \) where \( F(t) = \int_0^t f \).


3. Existence theorems. We prove the following

**Theorem 3.1.** \( f(t, y) \) is continuous in \( y \) for \( t \) a.e. on \( I \).

H2. \( f(t, y(t)) \subseteq P \) on \( I \) for \( y \) continuous on \( I \).

H3. \( f(t, y(t)) \geq g(t) \) a.e. on \( I \) where \( g \subseteq P \) on \( I \).

H4. \( |f(t, y(t)) - f(t, y^*(t))| \leq v(t) |y(t) - y^*(t)| \) a.e. on \( I \) where \( v \subseteq P \) (and hence in \( \mathcal{L} \)) on \( I \). Then there exists for the i.v.p. (1), a Picard sequence, which yields a solution, \( \psi(t) \), which is continuous and locally absolutely continuous (LAC) on \( I \), only if the sequence \( \{ \int_0^t (f_n - g) \} \) is EAC (equiabsolutely continuous) on \( I \).

**Lemma 3.1.** If

H1. \( f_n \subseteq P \) on \( I \) for each counting number \( n \).

H2. \( \lim_n f_n(t) = f(t) \) a.e. on \( I \).

H3. \( f_n(t) \geq g(t) \) a.e. on \( I \) for each \( n \) where \( g \subseteq P \) on \( I \). Then, \( f \subseteq P \) on \( I \), and \( \lim_n \int_0^t f_n = \int_0^t f \) only if the sequence \( \{ \int_0^t (f_n - g) \} \) is EAC on \( I \).

The proof of this lemma (see [7]) is based on a corresponding theorem by Vitali [11] for functions integrable in the Lebesgue sense. Briefly, by Theorem 2.2, \( (f_n - g) \subseteq \mathcal{L} \) on \( I \) and from [11]

\[
\lim_n (\mathcal{L}) \int_0^t [f_n - g] = (\mathcal{L}) \int_0^t [f - g].
\]

Consequently, \( (f - g) \subseteq P \), and \( \lim_n \int_0^t [f_n - g] = \int_0^t [f - g] \) and since \( g \subseteq P \) on \( I \), then \( f \subseteq P \) on \( I \) and \( \lim_n \int_0^t f_n = \int_0^t f \).

**Proof of Theorem 3.1.** Let \( y_0(t) = \alpha \);

\[
(2) \quad y_{n+1}(t) = \int_{t_0}^t f(s, y_n(s)) ds + \alpha \quad (n = 0, 1, 2, \cdots).
\]

We note that \( y_1(t) \) is continuous and LAC (see Saks [10, p. 251]) on \( I \). By induction, we have that \( y_n(t) \) is continuous and LAC on \( I \) for each counting number \( n \). Define

\[
(3) \quad u_n(t) = y_{n+1}(t) - y_n(t) \quad (n = 0, 1, 2, \cdots).
\]

Then, \( u_0(t) = \int_{t_0}^t f(s, \alpha) ds \) and
Under $H_4$, the integral in (4) may be taken in the Lebesgue sense. Now, since $u_0(t)$ is continuous on $I$, then there exists a number $k$ such that for $t$ on $I$, $|u_0(t)| < k$ and $|u_1(t)| < k f_0 v$ and in general, since $v(t) \geq 0$ a.e. on $I$

$$|u_n(t)| < k \left[ \int_0^t v \right]^n / n! \leq k \left[ \int_0^1 v \right]^n / n!.$$  

Consequently, $\sum_{i=0}^n |u_i(t)|$ converges uniformly on $I$. But, $\sum_{i=0}^n u_i(t) = y_{n+1}(t) - y_0(t)$. Hence, there exists a function $\psi(t)$ such that $\lim_n y_n(t) = \psi(t)$ uniformly on $I$ where $\psi(t)$ is continuous and LAC on $I$. From $H_1$ we have $\lim_n f(t, y_n(t)) = f(t, \psi(t))$ a.e. on $I$.

Then, $H_3$, and Lemma 3.1, yield

$$\psi(t) = \int_{t_0}^t f(s, \psi(s)) ds + \alpha$$

where $\psi(t_0) = \alpha$ and $\psi'(t) = f(t, \psi(t))$ a.e. on $I$. Q.E.D.

Under the hypotheses $H_1$–$H_3$ of Theorem 3.1, and by use of the Cauchy-Euler method, Northcutt [9] showed that there exists a function $\psi(t)$, continuous and LAC, which is a solution of the i.v.p. (1) only if the sequence $\{f_n \Delta_n g\}$ is EAC on $I$. His method of proof is based on Ascoli's theorem on a uniformly bounded set of equicontinuous functions on $I$, and on Lemma 3.1. Uniqueness is not to be expected in this case.

4. Uniqueness theorems.

**Theorem 4.1.** The solution $\psi(t)$ in Theorem 3.1 is unique.

**Proof.** Assume that there exists for the i.v.p. (1) another solution $\psi^*(t)$. Let $Y(t) = \psi(t) - \psi^*(t)$ for $t$ on $I$. Then,

$$Y(t) = \int_{t_0}^t \left[ f(s, \psi(s)) - f(s, \psi^*(s)) \right] ds$$

and

$$0 \leq |Y(t)| \leq \lim_n k \left[ (\mathcal{L}) \int_0^1 v \right]^n / n!.$$  

Hence $Y(t) = 0$ for $t$ on $I$ and uniqueness of solution for (1) follows.

The following involve generalizations of some of the results of Ettlinger [2].
**Theorem 4.2.** If \( \psi_1(t) \) and \( \psi_2(t) \) are LAC on \( I \) and satisfy the i.v.p.
(1) in a region \( R = \{(t, y) | 0 \leq t \leq 1, \text{ all } y\} \). If further, \( \psi_1(t) \) and \( \psi_2(t) \) satisfy H2 and H4 of Theorem 3.1 on \( I \). Then, \( \psi_1(t) = \psi_2(t) \) on \( I \).

A proof of this theorem may be obtained by use of the following lemma, which is a generalization of Gronwall's lemma [4].

**Lemma 4.2.1.** If \( h \in L \) and \( g \in P \) on \( I \), \( x \) is LAC on \( I \), and satisfy the differential inequality
(6) \[ x'(t) + h(t)x(t) \leq g(t), \text{ a.e. on } I. \]

Then, for \( 0 \leq t_0 < t \leq 1 \),
(7) \[ x(t) \leq \exp \left(- \int_{t_0}^{t} h \right) \left[ \int_{t_0}^{t} g \exp \left( \int_{t_0}^{s} h \right) + x(t_0) \right]. \]

**Proof.** Since \( \exp(\int_{t_0}^{t} h) > 0 \) we have
(8) \[ x'(t) \exp \left( \int_{t_0}^{t} h \right) + h(t)x(t) \exp \left( \int_{t_0}^{t} h \right) \leq g(t) \exp \left( \int_{t_0}^{t} h \right). \]

Now, \( g \in P \) and \( \exp(\int_{t_0}^{t} h) \) is absolutely continuous on \( I \). Then, by Theorem 2.2, \( g \cdot \exp(\int_{t_0}^{t} h) \in P \) on \( I \). Furthermore,
\[
\frac{d}{dt} \left[ x(t) \exp \left( \int_{t_0}^{t} h \right) \right] = x'(t) \exp \left( \int_{t_0}^{t} h \right) + h(t)x(t) \exp \left( \int_{t_0}^{t} h \right) \text{ a.e. on } I.
\]

Hence, by Corollary 2.1.1, and (8), we have for \( t \geq t_0 \)
(9) \[ \left[ x(s) \exp \left( \int_{t_0}^{s} h \right) \right]_0^{s} \leq \int_{t_0}^{s} g(s) \exp \left( \int_{t_0}^{s} h \right), \]
and statement (7) is thus obtained. We note that the equality in (7), \( \phi(t) = \exp(-\int_{t_0}^{t} h) \left[ \int_{t_0}^{t} g \exp(\int_{t_0}^{s} h) + \phi(t_0) \right] \), represents a solution for the linear differential equation \( x'(t) + h(t)x(t) = g(t) \).

**Corollary 4.2.1.** If \( x(t_0) = 0 \) and, a.e. on \( I \), \( g(t) = 0 \) and \( x(t) \geq 0 \), then \( x(t) = 0 \) on \([t_0, 1]\).

**Proof of Theorem 4.2.** Let \( x(t) = |\psi_1(t) - \psi_2(t)| \) on \( I \). Then, \( x'(t) \leq v(t)x(t) \) a.e. on \( I \). Since \( x(t_0) = 0 \), then for \( t \leq t_0 \) and by Corollary 2.1.1
(10) \[ x(t) \geq \int_{t_0}^{t} v x. \]
Now, from H4 and by Corollary 2.1.2, \(|f(t, \psi_1(t)) - f(t, \psi_2(t))| \subseteq P\) on \(I\), and is also integrable in the Lebesgue sense. Hence, for \(t\) on \(I\), H4 yields

\[
x(t) \leq \int_{t_0}^{t} vx.
\]

From (10) and (11) we have on \([0, t_0]\)

\[
x(t) = \int_{t_0}^{t} vx
\]

and \(x'(t) = v(t)x(t)\) a.e. on \([0, t_0]\). Corollary 4.2.1 then yields

\[
x(t) = 0 \quad \text{on } I.
\]

Hence, \(\psi_1(t) = \psi_2(t)\) on \(I\).

5. Remark. In equation (1), \(y(t)\) may be considered as a real valued vector function defined in a Euclidean space of \(n\) dimensions. Conclusions follow as in the scalar case.

References


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