BOUNDS FOR ZEROS OF SOME SPECIAL FUNCTIONS

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ABSTRACT. For $n \geqq 1$ let b_n and c_n be zeros (ordered by increasing values) of $u(x)$ and $v(x)$, respectively, which are nontrivial solutions of $u''+p(x)u=0$ and $v''+q(x)v=0$ with continuous $p(x)$ and $q(x)$. It is shown that if $b_n-c_n \to 0$ as $n \to \infty$, $p(x) \geqq q(x)$, and either $p(x)$ or $q(x)$ is nonincreasing, then $b_n \geqq c_n$ for $n \geqq 1$. Inequalities related to asymptotic expansions are obtained for the negative zeros a_n of the Airy function $Ai(z)$ and the zeros $j_{\nu,n}$ of the Bessel function $J_\nu(x)$.

The principal theorem, which gives an inequality for zeros of solutions of linear second order differential equations, is proved by induction using the Sturm comparison theorem at each step. This procedure differs from previous applications of the Sturm comparison theorem to orthogonal polynomials [**3**, pp. 120–130] and Bessel functions [**4**, pp. 518–521] since the common zero of the solutions of the differential equations is at infinity here, i.e., approached asymptotically.

THEOREM 1. *For $n \geqq 1$ let b_n and c_n be zeros (ordered by increasing values) of $u(x)$ and $v(x)$, respectively, which are nontrivial solutions of $u''+p(x)u=0$ and $v''+q(x)v=0$ with continuous $p(x)$ and $q(x)$. If $b_n-c_n \to 0$ as $n \to \infty$, $p(x) \geqq q(x)$, and either $p(x)$ or $q(x)$ is nonincreasing, then $b_n \geqq c_n$ for $n \geqq 1$.*

PROOF. The zeros b_n and c_n are simple and cannot have a finite accumulation point [**1**, pp. 223–225]. For $p(x) \equiv q(x)$, the theorem is true since $b_n = c_n$. Assume $p(x) \not\equiv q(x)$ and $b_n < c_n$ for some n. Let k be the least positive integer such that $b_k < c_k$. If we let $d_n = c_n - b_n$, then $d_k > 0$. We will show by induction that $d_n \geqq d_k > 0$ for all $n \geqq k$. Assume $d_m \geqq d_k > 0$ for some $m \geqq k$. If $w(x) = v(x+d_m)$, then $w''(x)+q(x+d_m)w(x) = 0$, $w(b_m)=0$, and $w(c_{m+1}-d_m)=0$. Since either $p(x) \geqq p(x+d_m) \geqq q(x+d_m)$ or $p(x) \geqq q(x) \geqq q(x+d_m)$, the Sturm comparison theorem implies $b_m < b_{m+1} < c_{m+1} - d_m$. Thus $c_{m+1} - b_{m+1} = d_{m+1} > d_m \geqq d_k > 0$, which completes our induction proof. However, $c_n - b_n = d_n \geqq d_k > 0$ for all $n \geqq k$ contradicts $b_n - c_n \to 0$. Consequently, no least positive

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integer k exists such that $b_k < c_k$ so that we must have $b_n \geqq c_n$ for all positive integers n.

THEOREM 2. *If a_n is the nth negative zero of the Airy function $Ai(z)$, then for $n \geqq 1$,*

$$-\left[\frac{3\pi}{8}(4n-1)+\frac{3}{2}\arctan\frac{5}{18\pi(4n-1)}\right]^{2/3} \leqq a_n \leqq -\left[\frac{3\pi}{8}(4n-1)\right]^{2/3}.$$

PROOF. It is well known [2, p. B48] that $a_n + [3\pi(4n-1)/8]^{2/3} \to 0$ as $n \to \infty$. The function $u(\zeta) = \pi^{1/2}x^{1/4}Ai(-x)$ where $\zeta = 2x^{3/2}/3$ satisfies $u'' + (1+5/(36\zeta^2))u = 0$ and $v(\zeta) = \cos(\zeta - \pi/4)$ satisfies $v'' + v = 0$. From Theorem 1 we have $2(-a_n)^{3/2}/3 \geqq n\pi - \pi/4$ from which the right-hand inequality follows immediately.

To prove the left-hand inequality we let

$$u(\zeta) = \frac{[(72)^2\zeta^2 + 25]^{1/2}}{[(72)^2\zeta^2 + 385]^{1/2}} \cos\left(\zeta - \pi/4 - \arctan\frac{5}{72\zeta}\right),$$

which satisfies

$$u'' + \left(1 + \frac{5}{36\zeta^2} + \frac{385[206(72)^2\zeta^2 - 3850]}{72\zeta^2[(72)^2\zeta^2 + 385]^2}\right)u = 0.$$

If we now apply Theorem 1 with $v(\zeta) = \pi^{1/2}x^{1/4}Ai(-x)$, then the already established inequality implies $206(72)^2\zeta^2 > 3850$. Consequently, $2(-a_n)^{3/2}/3 \leqq b_n$ where

$$b_n - \pi/4 - \arctan\frac{5}{72b_n} = n\pi - \pi/2.$$

The left-hand inequality now follows from

$$b_n \leqq n\pi - \pi/4 + \arctan(5/18\pi(4n-1)).$$

THEOREM 3. *If $j_{\nu,n}$ is the nth positive zero of the Bessel function $J_\nu(x)$ for $\nu \geqq 0$, then for $\nu \leqq 1/2$,*

$$n\pi + \frac{\nu\pi}{2} - \frac{\pi}{4} \leqq j_{\nu,n} \leqq n\pi + \frac{\nu\pi}{2} - \frac{\pi}{4} - \frac{4\nu^2 - 1}{8\left(n\pi + \frac{\nu\pi}{2} - \frac{\pi}{4}\right)};$$

and for $\nu \geqq 1/2$, $j_{\nu,n} \leqq n\pi + \nu\pi/2 - \pi/4$.

PROOF. The inequalities in the above theorem are related to McMahon's asymptotic expansion of $j_{\nu,n}$ [4, p. 506]. The proof follows from Theorem 1 when the following functions and differential equations are considered with $\alpha=(4\nu^2-1)/8$. The function $(\pi x/2)^{1/2}J_\nu(x)$ satisfies $u''+(1-2\alpha/x^2)u=0$, and $\cos(x-\nu\pi/2-\pi/4)$ satisfies $v''+v=0$. The function

$$(1-\alpha/x^2)^{-1/2}\cos(x-\nu\pi/2-\pi/4+\alpha/x)$$

satisfies

$$w''+\left(1-\frac{2\alpha}{x^2}+\frac{\alpha^2(x^2-\alpha)^2-3\alpha x^4}{x^4(x^2-\alpha)^2}\right)w=0.$$

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