A FLAT INTEGRAL FOR FUNCTIONALS DEFINED ON SAMPLE PATHS OF A BROWNIAN PROCESS WITH THE PARAMETER IN N-DIMENSIONAL SPACE

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In a recent paper [3], M. Pincus defined the flat integral for functionals defined on sample paths of general Gaussian processes and showed its application to Hammerstein integral equations. A specific flat integral of importance is obtained when the Gaussian process is the well-known Brownian process with the parameter in 1-dimensional space [3]. In this note, we consider a Brownian process with the parameter in N-dimensional space and derive a flat integral for functionals defined on sample paths of this process.

Let $X(t)$, $t=(t_1, \cdots, t_N)$, $0 \leqq t_i \leqq 1$, $i=1, \cdots, N$, be a Gaussian process with the parameter t in the N-dimensional Euclidean cube E_N, with mean 0, and positive definite covariance function

$$\rho(t, s) = E(X(t)X(s)) = (1/2) \min(t_1, s_1) \cdots \min(t_N, s_N). \tag{1}$$

This process is an extension to N dimensions of the Brownian process with 1-dimensional parameter since along any fixed coordinate, the process is Brownian motion with 1-dimensional parameter. A discussion of this process and its relation to the process of Kitagawa's functional integral [1] and the Wiener measure in N-dimensional space studied by J. Yeh [5] is given in [4]. It should be noted that the process is not Lévy's Brownian process with the parameter in N-dimensional space [2] which defines Brownian motion along any radial line from the origin.

Let $E_x^\rho\{\ \}$ denote the expectation on the process $X(t)$, $t \in E_N$, and let $G(\cdot)$ be a functional defined on sample paths x of $X(t)$ for which $E_x^\rho\{G(x)\}$ is defined. Let A be the positive definite Hilbert-Schmidt operator defined by the equation

$$(Ax)(t) = \int_{E_N} \rho(t, s)x(s)ds, \qquad x \in L^2, \tag{2}$$

where $\rho(t, s)$ is the covariance function given above, (1). According to the definition given by M. Pincus [3], if $A^{-1/2}$ exists then we write

$$E_x^\rho\{G(x)\} \text{ “=” } \int G(x) \exp(-(1/2)(A^{-1/2}x, A^{-1/2}x))\delta x. \tag{3}$$

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The symbol $\int \delta x$ is called the flat integral and the expression on the right of (3) is the flat integral of the functional $G(\cdot)$. The flat integral serves as a heuristic formalism for manipulating Gaussian expectations and makes theorems concerning such expectations transparent. In the following theorem, we give the expression for the flat integral for functionals defined on sample paths of the Brownian process $X(t)$ with t in N-dimensional space.

THEOREM. *Let $G(\cdot)$ be a functional defined on sample paths x of the process $X(t)$, $t \in E_N$ for which $E_x^o\{G(x)\}$ is defined. The flat integral of G is defined by the expression*

$$(4)\quad \int G(x) \exp\left(-\int_0^1 \cdots \int_0^1 \left(\frac{\partial^N x(t_1, \cdots, t_N)}{\partial t_1 \cdots \partial t_N}\right)^2 dt_1 \cdots dt_N\right) \delta x.$$

PROOF. Consider the case $N=2$. To obtain the proof, it suffices to show, according to (3), that

$$(5)\qquad (1/2)(A^{-1/2}x, A^{-1/2}x) = \int_0^1 \int_0^1 \left(\frac{\partial^2 x(t_1, t_2)}{\partial t_1 \partial t_2}\right)^2 dt_1 dt_2.$$

Using definitions (1) and (2), we have

$$(6)\quad \begin{aligned} \frac{\partial}{\partial t_1}(Ax)(t_1, t_2) &= \frac{1}{2}\frac{\partial}{\partial t_1}\int_0^1 \int_0^1 \min(t_1, s_1)\min(t_2, s_2) x(s_1, s_2) ds_1 ds \\ &= \frac{1}{2}\left\{\frac{\partial}{\partial t_1}\int_0^1 \int_0^{t_1} s_1 \min(t_2, s_2) x(s_1, s_2) ds_1 ds_2\right. \\ &\qquad \left. + \frac{\partial}{\partial t_1}\int_0^1 \int_{t_1}^1 t_1 \min(t_2, s_2) x(s_1, s_2) ds_1 ds_2\right\} \\ &= \frac{1}{2}\int_0^1 \int_{t_1}^1 \min(t_2, s_2) x(s_1, s_2) ds_1 ds_2. \end{aligned}$$

$$(7)\quad \frac{\partial^2}{\partial t_1^2}(Ax)(t_1, t_2) = -\frac{1}{2}\int_0^1 \min(t_2, s_2) x(t_1, s_2) ds.$$

$$(8)\quad \begin{aligned} &\frac{\partial^3}{\partial t_1^2 \partial t_2}(Ax)(t_1, t_2) \\ &\quad = -\frac{1}{2}\frac{\partial}{\partial t_2}\left\{\int_0^{t_2} s_2 x(t_1, s_2) ds_2 + \int_{t_2}^1 t_2 x(t_1, s_2) ds_2\right\} \\ &\quad = -\frac{1}{2}\int_{t_2}^1 x(t_1, s_2) ds_2. \end{aligned}$$

$$\frac{\partial^4}{\partial t_1^2 \partial t_2^2}(Ax)(t_1, t_2) = \tfrac{1}{2}x(t_1, t_2). \tag{9}$$

Therefore,

$$A^{-1} = 2\partial^4/\partial t_1^2 \partial t_2^2 \tag{10}$$

and equation (5) holds. The proof is easily generalized to arbitrary dimensions N.

References

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