ON THE ALGEBRAIC INDEPENDENCE OF
SYMMETRIC FUNCTIONS

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Abstract. The purpose of this note is to establish a necessary and sufficient condition for the algebraic independence of certain sets of homogeneous symmetric polynomials which is used in 2. to solve a problem proposed by L. Flatto [2].

1. We are concerned with the algebraic independence of a set of homogeneous symmetric polynomials in \( n \) variables with complex coefficients:

\[
h_1, h_2, \ldots, h_n \subseteq C[x_1, x_2, \ldots, x_n],
\]

where \( h_k \) is homogeneous of degree \( k \). By the fundamental theorem on symmetric polynomials each \( h_k \) can be written uniquely as a polynomial in the elementary symmetric polynomials \( s_1, s_2, \ldots, s_n \) of \( x_1, \ldots, x_n \). Since \( h_k \) is homogeneous of degree \( k \) it can even be written as a polynomial in \( s_1, s_2, \ldots, s_k \) which is linear in \( s_k \):

\[
h_k = G_k(s_1, s_2, \ldots, s_{k-1}) + c_k s_k,
\]

where \( G_k(0, 0, \ldots, 0) = 0 \) and \( c_k \in C. h_1, h_2, \ldots, h_n \) are algebraically independent over \( C \) if and only if each \( c_k \neq 0 \) for \( k = 1, 2, \ldots, n \). Indeed if each \( c_k \neq 0 \) the system (1) can be solved recursively for \( s_1, s_2, \ldots, s_n \) in terms of the \( h \)'s, and \( s_1, s_2, \ldots, s_n \) are algebraically independent over \( C \). If not all \( c_k \neq 0 \) let \( c_i \) be the first vanishing \( c \). For \( i = 1 \) we have \( h_1 = 0 \). For \( i > 1 \) we can solve the partial system

\[
h_k = G_k(s_1, s_2, \ldots, s_{k-1}) + c_k s_k, \quad k = 1, 2, \ldots, i-1 \text{ for } s_1, s_2, \ldots, s_{i-1}
\]

in terms of \( h_1, h_2, \ldots, h_{i-1} \) and enter these solutions into \( h_i = G_i(s_1, s_2, \ldots, s_{i-1}) \).

It is possible to determine the coefficients \( c_k \) without knowing the actual representation (1). For \( k \leq n \) let \( \omega_k \) denote a primitive \( k \)th root of unity. Then \( \omega_k, \omega_k^2, \ldots, \omega_k^k \) is the complete set of zeros of the polynomial \( x^k - 1 \) and we have

\[
s_i(\omega_k, \omega_k^2, \ldots, \omega_k^k, 0, \ldots, 0) = 0 \quad \text{for } i < k,
\]

\[
s_k(\omega_k, \omega_k^2, \ldots, \omega_k^k, 0, \ldots, 0) = (-1)^{k-1},
\]

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We have proved

**Theorem 1.** Let $h_k$ be a homogeneous symmetric function of degree $k$ in $x_1, x_2, \ldots, x_n$ with complex coefficients ($k = 1, 2, \ldots, n$). $h_1, h_2, \ldots, h_n$ are algebraically independent over $\mathbb{C}$ if and only if

$$h_k(\omega_k, \omega_k, \ldots, \omega_k, 0, \ldots, 0) \neq 0$$

for $k = 1, 2, \ldots, n$, where $\omega_k$ is a primitive $k$th root of unity.

2. We apply Theorem 1 to the solution of a problem of L. Flatto [2] and prove

**Theorem 2.** Let $P_{2k}(x) = \sum_{\pm} (\pm x_1 \pm x_2 \pm \cdots \pm x_n)^{2k}$. $P_2(x), P_4(x), \ldots, P_{2n}(x)$ are algebraically independent.

Direct computation gives

$$P_{2k}(x) = 2^n \sum_{\lambda_1+\lambda_2+\cdots+\lambda_n=k} \frac{(2k)!}{(2\lambda_1)!(2\lambda_2)! \cdots (2\lambda_n)!} x_1^{2\lambda_1} x_2^{2\lambda_2} \cdots x_n^{2\lambda_n}.$$

Let $y_i = x_i^2$ for $i = 1, 2, \ldots, n$. Then $P_{2k}(x) = h_k(y)$ is a homogeneous symmetric function of degree $k$ in $y_1, \ldots, y_n$. According to Theorem 1 we must show

$$\sum_{\lambda_1+\lambda_2+\cdots+\lambda_n=k} \frac{1}{\omega_k^{\lambda_1+2\lambda_2+\cdots+k\lambda_k}} \neq 0.$$

We call the $k$-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ a vector of exponents. The “length” of a vector of exponents be the number of its nonzero components. Let $\{\nu_1, \nu_2, \ldots, \nu_r\}$ be a set of $r \leq k$ positive integers satisfying $\sum \nu_i = k$. We determine all vectors of exponents of length $r$ obtained by assigning $\nu_1, \nu_2, \ldots, \nu_r$ to $r$ different components and compute the sum $S_k(\nu_1, \nu_2, \ldots, \nu_r)$ of the corresponding powers of $\omega_k$:

$$S_k(\nu_1, \nu_2, \ldots, \nu_r) = \sum_{m_1=1}^{k} \omega_k^{m_1 \nu_1} \cdot \sum_{m_2=1}^{k} (1 - \delta_{m_1m_2}) \omega_k^{m_2 \nu_2} \cdots \sum_{m_r=1}^{k} \prod_{i=1}^{r-1} (1 - \delta_{m_im_r}) \omega_k^{m_r \nu_r}.$$

This equation is clearly true for $r = 1, \nu_1 = k$. Assume now that $r > 1$. Since $\nu_r < k$ we have $\sum_{m_r} \omega_k^{m_r \nu_r} = 0$ and $\sum_{m_r} \prod_{i < r} (1 - \delta_{m_im_r}) \omega_k^{m_r \nu_r} = -\sum_{i < r} \omega_k^{m_i \nu_i}$ giving

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\[ S_k(v_1, \ldots, v_r) = - \sum_{i=1}^{r-1} S_k(v_1 + \delta_1 v_r, v_2 + \delta_2 v_r, \ldots, v_{r-1} + \delta_{r-1} v_r) \]

\[ = (-1)(r - 1) \cdot (-1)^{r-2} \cdot k(r - 2)! \]

by induction.

To obtain the sum of all terms in (2) which have the common coefficient \((2^n_1) \cdot \cdots \cdot (2^n_r))^{-1}\) we must divide \(S_k(v_1, v_2, \ldots, v_r)\) by \(n_1! n_2! \cdots n_k!\) where \(n_i\) is the number of times the integer \(i\) occurs in the set \(\{v_1, v_2, \ldots, v_r\}\). Using

\[ \sum_{i=1}^{k} n_i = r \]

we get

\[ \sum_{\lambda_1 + \lambda_2 + \cdots + \lambda_k = k} \frac{1}{(2\lambda_1)! \cdots (2\lambda_k)!} \left( \sum_{l=1}^{\lambda_1} \frac{n_l - 1}{n_1! n_2! \cdots n_k!} \right) \frac{1}{(2l)! (4l)! n_2 \cdots (2k)! n_k} \]

(3)

Now we apply to the right-hand side of (3) the following result of Cauchy [1]: *The formal equation*

\[ b_1 z + \frac{1}{2} b_2 z^2 + \frac{1}{3} b_3 z^3 + \cdots = \log(1 + a_1 z + a_2 z^2 + \cdots) \]

gives rise to the identity

\[ b_k = k \sum_{\sum \lambda_i = k} (-1)^{n_{l-1}} \frac{\left( \sum n_l - 1 \right)!}{n_1! n_2! \cdots n_k!} \cdot a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}. \]

By letting \(a_l = 1/(2l!)\) for \(l = 1, 2, \ldots, k\) we obtain the right-hand side of (3). It is convenient to replace in (4) the variable \(z\) by \(t^2\) and to differentiate with respect to \(t\). Then

\[ \sum_{l=0}^{\infty} \frac{z^t}{2l!} = \sum_{l=0}^{\infty} \frac{t^{2l}}{2l!} = \frac{1}{2} (e^t + e^{-t}), \]

and we have

\[ \sum_{l=1}^{\infty} 2b_l t^{2l-1} = \frac{d}{dt} \log \left( \frac{e^t + e^{-t}}{2} \right) = \tanh(t). \]

We obtain the value of (2) by dividing 2 into the \((2k - 1)\)st coefficient of the Taylor expansion of \(\tanh(t)\). According to [3, pp. 298–299], this coefficient equals
\[
\frac{2^{2k}(2^{2k} - 1)B_{2k}}{(2k)!}, \quad B_{2k} = (-1)^{k-1} \frac{2(2k)!}{(2\pi)^{2k}} \sum_{l=1}^{\infty} \frac{1}{l^{2k}}
\]

and is always different from zero. This establishes the theorem.

**Bibliography**


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\(^1\) Operated by Union Carbide Corporation for the U. S. Atomic Energy Commission.