ON THE ALGEBRAIC INDEPENDENCE OF SYMMETRIC FUNCTIONS

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ABSTRACT. The purpose of this note is to establish a necessary and sufficient condition for the algebraic independence of certain sets of homogeneous symmetric polynomials which is used in 2. to solve a problem proposed by L. Flatto [2].

1. We are concerned with the algebraic independence of a set of homogeneous symmetric polynomials in n variables with complex coefficients:

$$h_1, h_2, \cdots, h_n \in C[x_1, x_2, \cdots, x_n],$$

where h_k is homogeneous of degree k. By the fundamental theorem on symmetric polynomials each h_k can be written uniquely as a polynomial in the elementary symmetric polynomials $s_1, s_2, \cdots, s_n$ of $x_1, \cdots, x_n$. Since h_k is homogeneous of degree k it can even be written as a polynomial in $s_1, s_2, \cdots, s_k$ which is linear in s_k:

$$h_k = G_k(s_1, s_2, \cdots, s_{k-1}) + c_k \cdot s_k, \tag{1}$$

where $G_k(0, 0, \cdots, 0) = 0$ and $c_k \in C$. *$h_1, h_2, \cdots, h_n$ are algebraically independent over C if and only if each $c_k \neq 0$ for $k = 1, 2, \cdots, n$.* Indeed if each $c_k \neq 0$ the system (1) can be solved recursively for $s_1, s_2, \cdots, s_n$ in terms of the h's, and $s_1, s_2, \cdots, s_n$ are algebraically independent over C. If not all $c_k \neq 0$ let c_i be the first vanishing c. For $i = 1$ we have $h_1 = 0$. For $i > 1$ we can solve the partial system $h_k = G_k(s_1, s_2, \cdots, s_{k-1}) + c_k s_k$, $k = 1, 2, \cdots, i-1$ for $s_1, s_2, \cdots, s_{i-1}$ in terms of $h_1, h_2, \cdots, h_{i-1}$ and enter these solutions into $h_i = G_i(s_1, s_2, \cdots, s_{i-1})$.

It is possible to determine the coefficients c_k without knowing the actual representation (1). For $k \leqq n$ let ω_k denote a primitive kth root of unity. Then $\omega_k, \omega_k^2, \cdots, \omega_k^k$ is the complete set of zeros of the polynomial $x^k - 1$ and we have

$$s_i(\omega_k, \omega_k^2, \cdots, \omega_k^k, 0, \cdots, 0) = 0 \qquad \text{for } i < k,$$
$$s_k(\omega_k, \omega_k^2, \cdots, \omega_k^k, 0, \cdots, 0) = (-1)^{k-1},$$

and

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$$h_k(\omega_k, \overset{2}{\omega_k}, \cdots, \overset{k}{\omega_k}, 0, \cdots, 0) = (-1)^{k-1} c_k.$$

We have proved

THEOREM 1. *Let h_k be a homogeneous symmetric function of degree k in $x_1, x_2, \cdots, x_n$ with complex coefficients ($k=1, 2, \cdots, n$). $h_1, h_2, \cdots, h_n$ are algebraically independent over C if and only if*

$$h_k(\omega_k, \overset{2}{\omega_k}, \cdots, \overset{k}{\omega_k}, 0, \cdots, 0) \neq 0$$

for $k=1, 2, \cdots, n$, where ω_k is a primitive kth root of unity.

2. We apply Theorem 1 to the solution of a problem of L. Flatto [2] and prove

THEOREM 2. *Let $P_{2k}(x) = \sum_{\pm} (\pm x_1 \pm x_2 \pm \cdots \pm x_n)^{2k}$. $P_2(x)$, $P_4(x), \cdots, P_{2n}(x)$ are algebraically independent.*

Direct computation gives

$$P_{2k}(x) = 2_n \sum_{\lambda_1+\lambda_2+\cdots+\lambda_n=k} \frac{(2k)!}{(2\lambda_1)!(2\lambda_2)!\cdots(2\lambda_n)!} x_1^{2\lambda_1} x_2^{2\lambda_2} \cdots x_n^{2\lambda_n}.$$

Let $y_i = x_i^2$ for $i=1, 2, \cdots, n$. Then $P_{2k}(x) = h_k(y)$ is a homogeneous symmetric function of degree k in $y_1, \cdots, y_n$. According to Theorem 1 we must show

$$(2) \qquad \sum_{\lambda_1+\lambda_2+\cdots+\lambda_k=k} \frac{1}{(2\lambda_1)!(2\lambda_2)!\cdots(2\lambda_k)!} \omega_k^{\lambda_1+2\lambda_2+\cdots+k\lambda_k} \neq 0.$$

We call the k-tuple $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ a vector of exponents. The "length" of a vector of exponents be the number of its nonzero components. Let $\{\nu_1, \nu_2, \cdots, \nu_r\}$ be a set of $r \leqq k$ positive integers satisfying $\sum \nu_i = k$. We determine all vectors of exponents of length r obtained by assigning $\nu_1, \nu_2, \cdots, \nu_r$ to r different components and compute the sum $S_k(\nu_1, \nu_2, \cdots, \nu_r)$ of the corresponding powers of ω_k:

$$S_k(\nu_1, \nu_2, \cdots, \nu_r)$$

$$= \sum_{m_1=1}^{k} \omega_k^{m_1\nu_1} \cdot \sum_{m_2=1}^{k} (1 - \delta_{m_1m_2}) \omega_k^{m_2\nu_2} \cdot \cdots \cdot \sum_{m_r=1}^{k} \prod_{i=1}^{r-1} (1 - \delta_{m_im_r}) \omega_k^{m_r\nu_r}$$

$$= \sum_{m_1, m_2, \cdots, m_r}^{k} \prod_{i<j} (1 - \delta_{m_im_j}) \omega_k^{m_1\nu_1+m_2\nu_2+\cdots+m_r\nu_r} = (-1)^{r-1} k(r-1)!.$$

This equation is clearly true for $r=1$, $\nu_1 = k$. Assume now that $r>1$. Since $\nu_r < k$ we have $\sum_{m_r} \omega_k^{m_r\nu_r} = 0$ and $\sum_{m_r} \prod_{i<r} (1 - \delta_{m_im_r}) \omega_k^{m_r\nu_r} = -\sum_{i<r} \omega_k^{m_i\nu_r}$ giving

$$S_k(\nu_1, \cdots, \nu_r) = -\sum_{l=1}^{r-1} S_k(\nu_1 + \delta_{1l}\nu_r, \nu_2 + \delta_{2l}\nu_r, \cdots, \nu_{r-1} + \delta_{r-1,l}\nu_r)$$
$$= (-1)(r-1)\cdot(-1)^{r-2}\cdot k(r-2)!$$

by induction.

To obtain the sum of all terms in (2) which have the common coefficient $((2\nu_1)! \cdots (2\nu_r)!)^{-1}$ we must divide $S_k(\nu_1, \nu_2, \cdots, \nu_r)$ by $n_1!n_2! \cdots n_k!$ where n_i is the number of times the integer i occurs in the set $\{\nu_1, \nu_2, \cdots, \nu_r\}$. Using

$$\sum_{l=1}^{k} n_l = r$$

we get

$$\sum_{\lambda_1+\lambda_2+\cdots+\lambda_k=k} \frac{1}{(2\lambda_1)! \cdots (2\lambda_k)!} \omega^{\lambda_1+2\lambda_2+\cdots+k\lambda_k}$$
(3)
$$= \sum_{\Sigma l n_l=k} (-1)^{(\Sigma n_l - 1)} k \frac{(\sum n_l - 1)!}{n_1!n_2! \cdots n_k!} \cdot \frac{1}{(2!)^{n_1}(4!)^{n_2} \cdots (2k)!^{n_k}} .$$

Now we apply to the right-hand side of (3) the following result of Cauchy [1]: *The formal equation*

$$(4) \qquad b_1 z + \tfrac{1}{2} b_2 z^2 + \tfrac{1}{3} b_3 z^3 + \cdots = \log(1 + a_1 z + a_2 z^2 + \cdots)$$

gives rise to the identity

$$b_k = k \sum_{\Sigma l n_l = k} (-1)^{\Sigma n_l - 1} \frac{(\sum n_l - 1)!}{n_1!n_2! \cdots n_k!} a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k} .$$

By letting $a_l = 1/(2l!)$ for $l = 1, 2, \cdots, k$ we obtain the right-hand side of (3). It is convenient to replace in (4) the variable z by t^2 and to differentiate with respect to t. Then

$$\sum_{l=0}^{\infty} \frac{z^l}{2l!} = \sum_{l=0}^{\infty} \frac{t^{2l}}{2l!} = \frac{1}{2}(e^t + e^{-t}),$$

and we have

$$\sum_{i=1}^{\infty} 2b_i t^{2i-1} = \frac{d}{dt} \log\left(\frac{e^t + e^{-t}}{2}\right) = \tanh(t).$$

We obtain the value of (2) by dividing 2 into the $(2k-1)$st coefficient of the Taylor expansion of $\tanh(t)$. According to [**3**, pp. 298–299], this coefficient equals

$$\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}, \qquad B_{2k} = (-1)^{k-1}\frac{2(2k)!}{(2\pi)^{2k}}\sum_{l=1}^{\infty}\frac{1}{l^{2k}}$$

and is always different from zero. This establishes the theorem.

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[1] Operated by Union Carbide Corporation for the U. S. Atomic Energy Commission.