NOTE ON NONNEGATIVE MATRICES

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ABSTRACT. Let A be a nonnegative square matrix and $B = D_1AD_2$ where D_1 and D_2 are diagonal matrices with positive diagonal entries. Several proofs are known for the following theorem: If A is fully indecomposable then D_1 and D_2 can be chosen so that B is doubly stochastic. Moreover, D_1 and D_2 are unique up to a scalar factor. It is shown that these results can be easily obtained by considering a minimum of a certain rational function of several variables.

Several recent papers [1], [2], [3], [4] were devoted to the following problem: Given a nonnegative square matrix A, find the conditions for the existence of two diagonal matrices D_1 and D_2 such that D_1AD_2 is doubly stochastic. We shall show that it is related to a simple minimum problem. This leads to a short proof of Theorem (6.1) of [1] which avoids the use of Menon's operator.

We begin with some definitions. An $n\times n$ $(n\geqq 2)$ matrix A is reducible if there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

where A_1 is a $k\times k$ matrix, $1\leqq k\leqq n-1$. Otherwise we say that A is irreducible.

An $n\times n$ $(n\geqq 2)$ matrix A is fully indecomposable if there do not exist permutation matrices P and Q such that

$$PAQ = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

where A_1 is a $k\times k$ matrix, $1\leqq k\leqq n-1$.

THEOREM. *Let A be a nonnegative $n\times n$ fully indecomposable matrix. Then there exist diagonal matrices D_1 and D_2 with positive diagonals such that D_1AD_2 is doubly stochastic. Moreover D_1 and D_2 are uniquely determined up to scalar multiples.*

For the proof we need the following

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LEMMA. *Let A be a nonnegative $n\times n$ matrix. Then A is fully indecomposable if and only if there exist permutation matrices P and Q such that PAQ has a positive main diagonal and is irreducible.*

A short proof of this lemma appears in [1].

PROOF OF THE THEOREM. By the lemma we can assume that $A=(a_{ij})$ has positive main diagonal and is irreducible. Let

$$f(x_1, \cdots, x_n) = \prod_{k=1}^{n}\left(\sum_{i=1}^{n} a_{ki}x_i\right) \Big/ \prod_{k=1}^{n} x_k$$

the variables being restricted by

$$x_k > 0 \quad (1 \leqq k \leqq n), \qquad \sum_{k=1}^{n} x_k = 1. \tag{1}$$

Let (b_i) be a boundary point of the region (1) and, for instance, $b_1= \cdots =b_s=0$, $b_k>0$ $(s<k\leqq n)$. Since A is irreducible we infer that at least one entry $a_{ij}>0$ for $1\leqq i\leqq s$, $s<j\leqq n$. This implies that $f(x_1, \cdots, x_n)\rightarrow+\infty$ when $(x_k)\rightarrow(b_k)$. Therefore f attains its minimum in some point (c_k) of the region (1). The partial derivatives of f vanish at (c_k) since f is homogeneous. Hence,

$$\sum_{k=1}^{n} c_j a_{kj}\left(\sum_{i=1}^{n} a_{ki}c_i\right)^{-1} = 1 \qquad (j = 1, \cdots, n)$$

which proves the first assertion of the theorem.

For the uniqueness it is sufficient to prove the following assertion: If the matrices $X=(x_{ij})$, $D_1=\operatorname{diag}(d_1', \cdots, d_n')$, $D_2=\operatorname{diag}(d_1'', \cdots, d_n'')$ satisfy

(i) X is irreducible doubly stochastic with positive elements on the main diagonal;

(ii) $d_i'>0$, $d_i''>0$ $(1\leqq i\leqq n)$;

(iii) D_1XD_2 is doubly stochastic, then D_1 and D_2 are scalar matrices.

Since

$$\sum_{j=1}^{n} d_i' x_{ij} d_j'' = 1 \Rightarrow (\max d_i')(\min d_j'') \leqq 1,$$

$$\sum_{i=1}^{n} d_i' x_{ij} d_j'' = 1 \Rightarrow (\max d_i')(\min d_j'') \geqq 1.$$

we conclude that none of these inequalities is strict. This implies that $x_{rs}=0$ whenever $d_r'=\max d_i'$ and $d_s''>\min d_j''$ or $d_r'<\max d_i'$ and $d_s''=\min d_j''$. This contradicts (i) unless D_1 and D_2 are scalar matrices.

The proof is completed.

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