A PROPERTY OF TORSION-FREE MODULES OVER LEFT ORE DOMAINS

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ABSTRACT. It is well known that for an integral domain A, the property that a module is divisible if and only if it is injective is equivalent to the property that A is a Dedekind domain. In this paper, it is shown that if A is a left Ore domain, then a torsion-free left A-module is divisible if and only if it is injective.

Introduction. This paper will contain some results that grew out of an attempt to solve the following problem: Let A be a ring with no zero divisors and let E be a left A-module. When can E be embedded in a minimal divisible left A-module?

The problem has been solved in the case where A is a Dedekind domain with the result that any A-module can be embedded in a minimal divisible A-module. In the case where A is an integral domain, then it has been shown that every torsion-free A-module has a minimal divisible extension. This paper will show that over a left Ore domain, a torsion-free left A-module is divisible if and only if it is injective. From this fact, it will follow that any torsion-free left A-module can be embedded in a minimal divisible left A-module.

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In what follows, all rings will contain a unit and all modules will be unitary left modules. The usual definitions of torsion-free, divisible and injective have been extended to left modules, but coincide with the usual definitions over commutative rings.

THEOREM 1. *If A is a left Ore domain, then a torsion-free left A-module is divisible if and only if it is injective.*

PROOF. The proof of the sufficiency is identical with the proof in the case where A is a commutative ring and will be omitted here. For the proof of the necessary condition, let E be a torsion-free and divisible left A-module and consider the diagram

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$$\begin{array}{ccccc} 0 & \rightarrow & \mathfrak{A} & \xrightarrow{i} & A \\ & \phi & \downarrow & & \\ & & E & & \end{array}$$

where $\mathfrak{A}$ is a left ideal of A. Let $\alpha \in \mathfrak{A}$ and consider $\phi(\alpha)$. If $\phi(\alpha)=0$ for every $\alpha \in \mathfrak{A}$ then let $f \in \operatorname{Hom}(A, E)$ be such that $f:\beta \rightarrow 0$ for every $\beta \in A$. Trivially we see that in this case $f|\mathfrak{A}=\phi$ and thus E is injective. If $\phi(\alpha) \neq 0$, then since E is divisible, there is an element $x \in E$ such that $\alpha x=\phi(\alpha)$. Let $f \in \operatorname{Hom}(A, E)$ be such that $f: 1 \rightarrow x$. We must show that $f|\mathfrak{A}=\phi$. Let $\beta \in \mathfrak{A}$, $\beta \neq 0$ and let r, $s \in A-\{0\}$ be such that $r\alpha=s\beta$. This is possible since A is a left Ore domain [**5**]. Then $f(\alpha)=\phi(\alpha)$ so $rf(\alpha)=r\phi(\alpha)$ and this implies $f(r\alpha)=\phi(r\alpha)$. Thus $f(s\beta)=\phi(s\beta)$ so $sf(\beta)=s\phi(\beta)$ and $s(f(\beta)-\phi(\beta))=0$. Since E is torsion-free, we get that $f(\beta)=\phi(\beta)$. Thus $f|\mathfrak{A}=\phi$ so E is an injective left A-module.

THEOREM 2. *If A is a ring with no zero divisors, then the following are equivalent*:

(a) *A is a left Ore domain.*

(b) *Every torsion-free left A-module has a torsion-free injective hull.*

PROOF. (a)$\Rightarrow$(b). Let E be a torsion-free left A-module and let $H(E)$ denote the injective hull of E. Suppose $x \in H(E)$, $\alpha \in A$, and $\alpha x=0$, $\alpha \neq 0$. Let B be the submodule of $H(E)$ generated by x and let $\beta \in A$, $\beta \neq 0$. There exist elements r, $s \in A-\{0\}$ such that $r\alpha=s\beta$. Thus $r\alpha x=0$ so $s(\beta x)=0$ and this says that $\beta x \notin E$ since E is torsion-free. Since $\beta x \in B$ for every $\beta \in A$ and since $\beta x \notin E$, $\beta \neq 0$, we see that $B \cap E=0$. $H(E)$ is an essential extension of E so this implies that $B=0$. $1 \cdot x \in B$ so $1 \cdot x=x=0$. Thus $H(E)$ is torsion-free.

(b)$\Rightarrow$(a). Suppose that every torsion-free left A-module E has a torsion-free injective hull $H(E)$. A_s (A considered as a left module over itself) is torsion-free, so A_s has a torsion-free injective hull $H(A_s)$. Let a, b be nonzero elements of A. Since A is a domain, the map $f: xa \rightarrow x$ from $Aa \rightarrow H(A_s)$ is a left A-homomorphism, so extends to a map $f': A \rightarrow H(A_s)$. Let $f'(1)=a^{-1}$. Then $ba^{-1} \in H(A_s)$. There is some element $\beta \in A-\{0\}$ such that $\beta(ba^{-1}) \in A_s$, for otherwise ba^{-1} would generate a submodule of $H(A_s)$ whose intersection with A_s would be 0, contradicting the fact that $H(A_s)$ is an essential extension of A_s. Let $\beta(ba^{-1})=\alpha$. Then $\alpha \neq 0$ and we note that $\alpha(aa^{-1})=\alpha$ so $(\beta b)a^{-1}=(\alpha a)a^{-1}$. Since $H(A_s)$ is torsion-free, $\alpha a=\beta b$ and we get that A is a left Ore domain.

COROLLARY. *If A is a left Ore domain, then every torsion-free left A-module can be embedded in a minimal divisible left A-module.*

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