THE CONVERSE OF CAUCHY'S THEOREM FOR ARBITRARY RIEMANN SURFACES

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Abstract. In this paper, we prove a generalization of the converse of Cauchy's theorem which is valid for arbitrary hyperbolic Riemann surfaces. The tools used are the Kuramochi compactification and the concept of generalized normal component.

Let $\bar{R}$ be a compact bordered Riemann surface with border $\beta$. Royden [3] shows that if $f$ is a continuous complex valued function on $\beta$ and if $\int_{\beta} f\phi = 0$ for each differential $\phi$ which is analytic on $\bar{R}$, then $f$ can be extended to an analytic function $F$ on $R$.

1. Let $R$ now be an arbitrary hyperbolic Riemann surface with Kuramochi boundary $\Delta$. Let $u$ be a complex valued harmonic function on $R$ with finite Dirichlet integral. We shall say that $u$ possesses in a generalized sense a normal derivative on $\Delta$ when a complex measure $\mu$ exists on $\Delta$, which possesses the following property: If $f$ is a complex Dirichlet function on $R$, then for each quasicontinuous extension $f$ to $R^*$

$$\langle du, df \rangle = \int f d\mu;$$

$\mu$ is called the normal derivative of $u$.

From Folgesatz 17.27 in [2], it follows that the class of complex valued harmonic functions with finite Dirichlet integral which possess normal derivatives is dense in the class $HD$ of all complex valued harmonic functions with finite Dirichlet integral.

Let $\omega$ be a square integrable harmonic differential on $R$. Then $\omega$ can be written uniquely as $du + \alpha$ where $u \in HD$ and $\alpha \in \Gamma_{\Lambda}^*$ (for the definition of $\Gamma_{\Lambda}^*$ see Chapter V in [1]). We shall say that $\omega$ possesses a normal component on $\Delta$ if $u$ possesses a normal derivative on $\Delta$ and define the normal component of $\omega$ to be the normal derivative of $u$.

Let $\Gamma_{\Lambda}$ denote the space of differentials $\phi$ where $\phi$ is a square integrable analytic differential on $R$. We are now ready to state our generalization of Royden's converse of Cauchy's theorem.
Theorem. Let \( R \) be a hyperbolic Riemann surface with Kuramochi boundary \( \Delta \). Then if \( f \) is a continuous complex valued function on \( \Delta \) having the property that
\[
\int_{\Delta} f \, d\mu = 0
\]
for all complex measures \( \mu \) which are normal components of differentials in \( \Gamma_{\alpha} \), then there exists an analytic function \( F \) on \( R \) having the property that
\[
\lim_{a \to b} F(a) = f(b)
\]
for all regular points \( b \).

Proof. The set \( N = \{ \phi \mid \phi \in \Gamma_{\alpha}, \phi \text{ possesses a normal component } \mu \text{ on } \Delta \} \) is dense in \( \Gamma_{\alpha} \). Let \( F = H_f \) where \( H_f \) denotes the solution of the Dirichlet problem with boundary function \( f \). Then if \( b \) is a regular point of \( \Delta \), it follows from the definition of a regular point that
\[
\lim_{a \to b} F(a) = f(b).
\]
Now let \( \phi \in N \). Then \( \langle \phi, dF \rangle = \int_{\Delta} f \, d\mu = 0 \).

Hence \( dF \) is orthogonal to \( N \) and consequently \( \Gamma_{\alpha} \). Hence \( F \) is analytic.

If \( \overline{R} \) denotes a compact bordered Riemann surface with border \( \beta \), then it is known that the Kuramochi compactification of \( R \) is \( \overline{R} \) and consequently the Kuramochi boundary of \( R \) is \( \beta \). It follows that our theorem reduces to Royden's theorem for the case of a compact bordered Riemann surface.

References

