THE CONVERSE OF CAUCHY'S THEOREM FOR ARBITRARY RIEMANN SURFACES

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ABSTRACT. In this paper, we prove a generalization of the converse of Cauchy's theorem which is valid for arbitrary hyperbolic Riemann surfaces. The tools used are the Kuramochi compactification and the concept of generalized normal component.

Let $\overline{R}$ be a compact bordered Riemann surface with border β. Royden [3] shows that if f is a continuous complex valued function on β and if $\int_\beta f\phi=0$ for each differential ϕ which is analytic on $\overline{R}$, then f can be extended to an analytic function F on R.

1. Let R now be an arbitrary hyperbolic Riemann surface with Kuramochi boundary Δ. Let u be a complex valued harmonic function on R with finite Dirichlet integral. We shall say that u possesses in a generalized sense a normal derivative on Δ when a complex measure μ exists on Δ, which possesses the following property: If f is a complex Dirichlet function on R, then for each quasicontinuous extension f to R^*

$$\langle du, df\rangle = \int fd\mu;$$

μ is called the normal derivative of u.

From Folgesatz 17.27 in [2], it follows that the class of complex valued harmonic functions with finite Dirichlet integral which possess normal derivatives is dense in the class HD of all complex valued harmonic functions with finite Dirichlet integral.

Let ω be a square integrable harmonic differential on R. Then ω can be written uniquely as $du+\alpha$ where $u\in \mathrm{HD}$ and $\alpha\in\Gamma_{h0}^*$ (for the definition of Γ_{h0}^* see Chapter V in [1]). We shall say that ω possesses a normal component on Δ if u possesses a normal derivative on Δ and define the normal component of ω to be the normal derivative of u.

Let $\bar{\Gamma}_a$ denote the space of differentials $\bar{\phi}$ where ϕ is a square integrable analytic differential on R. We are now ready to state our generalization of Royden's converse of Cauchy's theorem.

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THEOREM. *Let R be a hyperbolic Riemann surface with Kuramochi boundary Δ. Then if f is a continuous complex valued function on Δ having the property that*

$$\int_{\Delta} f d\mu = 0$$

for all complex measures μ which are normal components of differentials in $\bar{\Gamma}_a$, then there exists an analytic function F on R having the property that

$$\lim_{a \to b} F(a) = f(b)$$

for all regular points b.

PROOF. The set $N = \{\Phi \mid \Phi \in \bar{\Gamma}_a,\ \Phi$ possesses a normal component μ on $\Delta\}$ is dense in $\bar{\Gamma}_a$. Let $F = H_f$ where H_f denotes the solution of the Dirichlet problem with boundary function f. Then if b is a regular point of Δ, it follows from the definition of a regular point that

$$\lim_{a \to b} F(a) = f(b).$$

Now let $\Phi \in N$. Then $\langle \Phi, dF \rangle = \int_{\Delta} f d\mu = 0$.

Hence dF is orthogonal to N and consequently $\bar{\Gamma}_a$. Hence F is analytic.

If $\bar{R}$ denotes a compact bordered Riemann surface with border β, then it is known that the Kuramochi compactification of R is $\bar{R}$ and consequently the Kuramochi boundary of R is β. It follows that our theorem reduces to Royden's theorem for the case of a compact bordered Riemann surface.

REFERENCES

1. L. V. Ahlfors and L. Sario, *Riemann surfaces*, Princeton Math. Series, no. 26, Princeton Univ. Press, Princeton, N. J., 1960. MR 22 #5729.

2. C. Constantinescu and A. Cornea, *Ideale Ränder Riemannscher Flächen*, Ergebnisse der Math. und ihrer Grenzgebiete, Band 32, Springer-Verlag, Berlin, 1963. MR 28 #3151.

3. H. L. Royden, *The boundary values of analytic and harmonic functions*, Math. Z. **78** (1962), 1–24. MR 25 #2190.

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