WEAK SEQUENTIAL COMPLETENESS IN SPACES OF OPERATORS[1]

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Throughout, X and Y will be nontrivial normed linear spaces over the field of complex numbers, E the space $\mathfrak{L}(X, Y)$ of continuous linear transformations from X into Y equipped with the usual norm, which we denote by μ, W a linear subspace of the dual Y' of Y, and ω the set of positive integers. The norm-closed unit ball in X, for example, is written S_X, and the canonical embedding ϕ from Y into W' is defined by $\phi(y)(w) = w(y)$. Each f belonging to the tensor product $X \otimes W$ determines a continuous linear functional on E since if $\sum_{i=1}^k x_i \otimes w_i$ is any representation of f then

$$|f(T)| = \left| \sum_{i=1}^{k} w_i(Tx_i) \right| \leqq \sum_{i=1}^{k} \|w_i\| \, \|x_i\|$$

for each $T \in S_E$. It is easy to verify that the restriction to the subspace $X \otimes W$ of the dual norm in E' is a cross norm ψ on $X \otimes W$. For each W, therefore, we let $F_W = X \otimes_\psi W$, where $\otimes_\psi$ denotes the tensor product under the cross norm ψ, and consider F_W as a subspace of E'. Also, we will consider $X' \otimes_\lambda Y$ canonically embedded in E, where λ is the least cross norm [6].

Following Dixmier [1], we define the characteristic of W to be the number

$$\nu(W) = \inf_{y \in Y} \sup_{w \in S_W} |w(y)|, \qquad \|y\| = 1.$$

Thus, $0 \leqq \nu(W) \leqq 1$. Letting W_0 be the annihilator of W in Y, it is evident that $W_0 = \{0\}$ if $\nu(W) > 0$ and that $W_0 = \{0\}$ if and only if the topology $\sigma(Y, W)$ is Hausdorff.

In the work below, we are concerned with theorems dealing with relationships between the space Y equipped with a topology $\sigma(Y, W)$ and the space E equipped with the corresponding topology $\sigma(E, F_W)$. Let $\nu(W) > 0$, and consider the family of seminorms $\{p_{x,w}\}$ defined by

$$p_{x,w}(T) = |w(Tx)| \qquad (w \in W, x \in X)$$

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for each $T \in E$. For the case $W = Y'$, this family of seminorms defines the weak operator topology [2] which is precisely $\sigma(E, F_{Y'})$. More generally, if W is a subspace of Y' for which $\nu(W) > 0$, the family of seminorms $\{p_{x,w}\}$ defines on E a topology which is weaker than the weak operator topology and which is precisely $\sigma(E, F_W)$. Thus, operator theorists who are not interested in tensor products and crossnorms will find our work more transparent if they will view the results in a weak operator type setting.

THEOREM. (i) *If $\nu(W) = 1$ and Y is $\sigma(Y, W)$-sequentially complete, then S_E is $\sigma(E, F_W)$-sequentially complete.*

(ii) *Let $\nu(W) > 0$ and W be norm-closed in Y'. If Y is $\sigma(Y, W)$-sequentially complete and X is a Banach space, then E is $\sigma(E, F_W)$-sequentially complete. Conversely, if each $\sigma(E, F_W)$-Cauchy sequence in $(X' \otimes_\lambda Y) \cap S_E$ is $\sigma(E, F_W)$-convergent to some $T \in E$, then Y is $\sigma(Y, W)$-sequentially complete.*

(iii) *Let Y be a barrelled normed linear space. Suppose $Y' | W$ is of countable Hamel dimension and W is $\sigma(Y', Y)$-sequentially dense in Y'. If $\nu(W) > 0$, Y is $\sigma(Y, W)$-sequentially complete, and X is a Banach space, then E is $\sigma(E, F_W)$-sequentially complete. Conversely, if every $\sigma(E, F_W)$-Cauchy sequence in $(X' \otimes_\lambda Y) \cap S_E$ is $\sigma(E, F_W)$-convergent to an element $T \in E$, then Y is $\sigma(Y, W)$-sequentially complete.*

PROOF. To prove (i) let $\{T_n\}$ be a $\sigma(E, F_W)$-Cauchy sequence in S_E. Then for each $x \in X$ the sequence $\{T_n x\}$ is $\sigma(Y, W)$-convergent in Y. We define a mapping T from X into Y by writing $Tx = y$ if and only if $y = \sigma(Y, W)$-lim $T_n x$. Obviously, T is linear. Since $\nu(W) = 1$,

$$\|Tx\| = \sup_{w \in S_W} \lim_n |w(T_n x)| \leqq \|x\|.$$

Thus, $T \in S_E$. Finally,

$$\begin{aligned} \lim_n \left(\sum_i x_i \otimes w_i \right) (T_n) &= \lim_n \sum_i w_i(T_n x_i) \\ &= \sum_i w_i(T x_i) \\ &= \left(\sum_i x_i \otimes w_i \right) (T) \end{aligned}$$

for each $\sum_i x_i \otimes w_i \in F_W$.

We proceed to prove the first assertion of (ii) by considering the seminorm p_W on Y defined by

$$p_W(y) = \sup_{w \in S_W} |w(y)| \qquad (y \in Y).$$

Since $\nu(W)>0$ and $p_W(y)\leqq\|y\|\leqq\nu(W)^{-1}p_W(y)$ for each $y\in Y$, p_W is a norm on Y equivalent to the original norm and $\|w\|=p_W^0(w)$ for each $w\in W$, where p_W^0 denotes the dual norm in $(Y, p_W)'$. Therefore, (W, p_W^0) has characteristic equal to one in $(Y, p_W)'$. Let E^1 and F_W^1 be the spaces obtained by replacing Y with (Y, p_W) and W with (W, p_W^0) in the definitions of E and F_W, respectively. Since E and F_W are linearly homeomorphic to E^1 and F_W^1, respectively, it suffices, in the light of (i), to prove that every $\sigma(E^1, F_W^1)$-Cauchy sequence in E^1 is norm-bounded in E^1.

Let $\{T_n\}$ be a $\sigma(E^1, F_W^1)$-Cauchy sequence in E^1. Then $\{T_n\}$ is $\sigma(E^1, F_W^1)$-bounded, whence $\sup_{n\in\omega}|w(T_nx)|<+\infty$ for each $x\in X$, $w\in W$. Fix $x\in X$ and vary w throughout W. Since the canonical embedding ϕ of (Y, p_W) into $(W, p_W^0)'$ is an isometry, the sequence of scalars $\{p_W(T_nx)\}$ is bounded for each $x\in X$ by the Banach-Steinhaus theorem. Since X is complete, a second application of the Banach-Steinhaus theorem implies that $\{T_n\}$ is norm-bounded in E^1. Thus, the first assertion of (ii) is proved.

The second assertions of (ii) and (iii) may be proved simultaneously after two remarks have been made. Let $\{y_n\}$ be a $\sigma(Y, W)$-Cauchy sequence in Y. Assuming the hypothesis of (ii) there exists $M_0>0$ such that $p_W(y_n)\leqq M_0$, $n\in\omega$, since (W, p_W^0) has characteristic equal to one in $(Y, p_W)'$ and is closed in $(Y, p_W)'$. Therefore, $\|y_n\|\leqq\nu(W)^{-1}M_0$, $n\in\omega$. Assuming the hypothesis of (iii) the sequence $\{y_n\}$ is again norm-bounded by a result of Levin and Saxon [**3**]. Thus, in either case, there exists $M>0$ such that $\|y_n\|\leqq M$, $n\in\omega$. Choose $x_0\in X$ and $x'\in X'$ such that $\|x_0\|=M$, $\|x'\|=1/\|x_0\|$, and $x'(x_0)=1$. If Z is the null space of x' in X and $[x_0]$ is the (closed) linear span of x_0 in X then $X=Z\oplus[x_0]$, whence each $x\in X$ can be uniquely written as $x=z_x+\alpha_x x_0$ for some $z_x\in Z$ and scalar α_x. Put $T_n=x'\otimes y_n$, $n\in\omega$, and note that $\mu(T_n)\leqq 1$. Since $T_nx=\alpha_x y_n$, for each $\sum x_i\otimes w_i\in F_W$

$$\left|\left(\sum_i x_i\otimes w_i\right)(T_n)-\left(\sum_i x_i\otimes w_i\right)(T_m)\right|$$
$$\leqq\sum_i|\alpha_{x_i}|\,|w_i(y_n)-w_i(y_m)|,$$

from which it follows that $\{T_n\}$ is $\sigma(E, F_W)$-Cauchy. By hypothesis, there exists $T\in E$ such that $\{T_n\}$ is $\sigma(E, F_W)$-convergent to T, and since $\{y_n\}$ converges to Tx_0 in the $\sigma(Y, W)$-topology, the second parts of (ii) and (iii) are proved.

Finally, we prove the first part of (iii). Given any $\sigma(E, F_W)$-Cauchy sequence $\{T_n\}$ in E, we define the linear mapping T as in the proof of (i). If $x \in X$ then $\{T_n x\}$ is $\sigma(Y, W)$-Cauchy and hence norm-bounded in Y [3]. By the Banach-Steinhaus theorem there exists $C>0$ such that $\mu(T_n) \leqq C$, $n \in \omega$, whence

$$\begin{aligned} \|Tx\| &\leqq \nu(W)^{-1} \sup_{w \in S_W} |w(Tx)| \\ &= \nu(W)^{-1} \sup_{w \in S_W} \lim_n |w(T_n x)| \\ &\leqq \nu(W)^{-1} C \|x\|. \end{aligned}$$

Hence, T is continuous and, as before, T is the $\sigma(E, F_W)$-limit of $\{T_n\}$. △

REMARK. Suppose X and Y are Banach spaces, and let γ be the greatest cross norm. It is known [6] that $(X \otimes_\gamma Y)' = \mathfrak{L}(X, Y')$ and thus that $\psi = \gamma$ in this setting. The topology $\sigma(\mathfrak{L}(X, Y'), X \otimes_\psi Y)$ is therefore the weak-star topology on $\mathfrak{L}(X, Y')$, whence it follows that $\mathfrak{L}(X, Y')$ and its norm-closed unit ball are $\sigma(\mathfrak{L}(X, Y'), X \otimes_\psi Y)$-sequentially complete. This can also be proven using (i) and the first assertion of (ii) in the theorem, independent of the duality under the cross norm γ.

COROLLARY. *Let $X' \hat{\otimes}_\lambda Y$ be the λ-completion of $X' \otimes_\lambda Y$ for a Banach space Y and λ^0 the dual norm in $(X' \hat{\otimes}_\lambda Y)'$.*

(i) *For every weakly sequentially complete Banach space Y, the Banach space $l_1 \hat{\otimes}_\lambda Y$ is sequentially complete in the $\sigma(l_1 \hat{\otimes}_\lambda Y, c \otimes_{\lambda^0} Y')$ topology. Thus, in particular, $l_1 \hat{\otimes}_\lambda l_1$* is *$\sigma(l_1 \hat{\otimes}_\lambda l_1, c \otimes_{\lambda^0} m)$-sequentially complete.*

(ii) *Let J be the canonical embedding from X into X'' and $X'' | JX$ the quotient space of X'' with JX equipped with the usual norm. For each Banach space X such that $X'' | JX$ is separable, the space $X' \hat{\otimes}_\lambda l_1$ is $\sigma(X' \hat{\otimes}_\lambda l_1, X \otimes_{\lambda^0} m)$-sequentially complete. Thus, $X \hat{\otimes}_\lambda l_1$ is $\sigma(X \hat{\otimes}_\lambda l_1, X' \otimes_{\lambda^0} m)$-sequentially complete for every reflexive space X.*

(iii) *For every Banach space X such that $X'' | JX$ is separable, the space $X' \hat{\otimes}_\lambda l_1$ is $\sigma(X' \hat{\otimes}_\lambda l_1, X \otimes_{\lambda^0} c_0)$-sequentially complete. Thus, $X \hat{\otimes}_\lambda l_1$ is $\sigma(X \hat{\otimes}_\lambda l_1, X' \otimes_{\lambda^0} c_0)$-sequentially complete for every reflexive space X.*

PROOF. It is known [2, p. 515, #35] that if Y is weakly sequentially complete then $\mathfrak{L}(c, Y)$ consists entirely of compact mappings. Since c' is isometrically isomorphic to l_1 and l_1 has the approximation property [5, p. 108], $\mathfrak{L}(c, Y) = l_1 \hat{\otimes}_\lambda Y$. Also, it is known [4] that if $X'' | JX$ is separable, then $\mathfrak{L}(X, l_1) = X' \hat{\otimes}_\lambda l_1$. Both (i) and (ii) now follow from part (ii) of the theorem. Finally, (iii) follows directly from the remark since here also $\mathfrak{L}(X, l_1) = X' \hat{\otimes}_\lambda l_1$. △

The relationship between $\nu(W)$ and $\nu(F_W)$ is precisely as one would expect.

PROPOSITION. *Let X and Y be normed linear spaces and W a subspace of Y'. Then $\nu(W)=\nu(F_W)$.*

PROOF. There exists $x'\in X'$ such that $\|x'\|=1$. If $y\in Y$ and $f=\sum x_i \otimes w_i \in S_{F_W}$, then

$$\left|\left(\sum x'(x_i)w_i\right)(y)\right| = \left|f(x'\otimes y)\right| \leqq \|y\|$$

so that $\sum x'(x_i)w_i\in S_W$. It follows that

$$\sup_{f\in S_{F_W}} \left|f(x'\otimes y)\right| \leqq \sup_{w\in S_W} \left|w(y)\right|.$$

Thus,

$$\begin{aligned}\nu(F_W) &\leqq \inf_{\|y\|=1;y\in Y} \sup_{f\in S_{F_W}} \left|f(x'\otimes y)\right| \\ &\leqq \inf_{\|y\|=1;y\in Y} \sup_{w\in S_W} \left|w(y)\right| = \nu(W).\end{aligned}$$

To prove the reverse inequality, we define

$$\mu_W(T) = \sup_{f\in S_{F_W}} \left|f(T)\right| \qquad (T\in E)$$

and note that

$$\begin{aligned}\mu_W(T) &\geqq \sup_{\|x\|=\|w\|=1} \left|x\otimes w(T)\right| \qquad (x\in X,\ w\in W) \\ &= \sup_{\|x\|=1}\ \sup_{\|w\|=1} \left|w(Tx)\right| \qquad (x\in X,\ w\in W) \\ &\geqq \nu(W) \sup_{\|x\|=1} \|Tx\| \qquad (x\in X) \\ &= \nu(W)\mu(T).\end{aligned}$$

Thus,

$$\nu(F_W) = \inf_{\mu(T)=1} \mu_W(T) \geqq \nu(W). \quad \triangle$$

An extension of the theorem to the quasicomplete case exists and is not difficult to obtain. The author wishes to acknowledge this comment and other useful comments and suggestions made by the referee.

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