A REAL ANALOGUE OF THE GELFAND-NEUMARK THEOREM

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ABSTRACT. Let A be a real Banach *-algebra enjoying the following three conditions: $\|x^*x\| = \|x^*\| \, \|x\|$, $Spx^*x \geqq 0$, and $\|x^*\| = \|x\|$ $(x \in A)$. It is shown, after Ingelstam, Palmer, and Behncke, as a real analogue of the Gelfand-Neumark theorem, that A is isometrically *-isomorphic onto a real C^*-algebra acting on a suitable real (or complex) Hilbert space. The converse is obvious.

The aim of this note is, as a real analogue of the Gelfand-Neumark theorem, to prove the following

THEOREM. *A real Banach *-algebra A is isometrically *-isomorphic onto a real C^*-algebra acting on a real (or complex) Hilbert space if and only if it satisfies the following three conditions*:

(1) $\|x^*x\| = \|x^*\| \, \|x\|$,
(2) $Spx^*x \geqq 0$, *and*
(3) $\|x^*\| = \|x\|$ $(x \in A)$.

The above theorem was conjectured explicitly by Rickart [**5**, p. 181] and proved by Ingelstam [**2**] (cf. also Palmer [**3**], [**4**] and Behncke [**1**]). Their proofs were based on complexification of a real Banach *-algebra. An alternative proof which we shall give in this note will be done by real *-representation on real Hilbert space and by complexification of a real Hilbert space.

Let A be a real Banach *-algebra satisfying the conditions stated in the theorem, and H the set of hermitian elements in A. Let R be the field of real numbers. In view of (2), the involution is hermitian. Put $\mu(h) = \sup(\lambda; \lambda$ a spectrum of $h)$ for all h in H. In view of (2), A is symmetric. In view of (3), the involution is continuous. So, we can make use of Rickart [**5**, Lemma 4.7.10] to get the sublinearity of μ on H, that is,

(i) $\mu(\alpha h) = \alpha\mu(h)$ and
(ii) $\mu(h+k) \leqq \mu(h) + \mu(k)$,

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where $0 \leqq \alpha \in R$, h, $k \in H$. Owing to the extension theorem of Hahn and Banach, for a fixed element a in A, there exists a real linear functional, say g, on H such that $g(h) \leqq \mu(h)$ $(h \in H)$ and such that $g((aa^*)^2) = \mu((aa^*)^2)$. Decompose $x = h + k$, where $h = (1/2)(x + x^*) \in H$ and $k = (1/2)(x - x^*)$ being skew adjoint. Put $f(x) = g(h)$ for all x in A. Since $\mu(-x^*x) \leqq 0$, we have $f(x^*x) \geqq 0$. Thus, f is a real state on A. It is easy to construct a *-representation real Hilbert space and a real *-representation ψ of A. Moreover, if $aa^* \neq 0$, $\psi(a) \neq 0$. Hence, $\{a;\, aa^* = 0\}$ is the *-radical of A, that is, the intersection of kernels of all real *-representations of A. In view of (1), the *-radical must be $\{0\}$. Thus, there exist a *-representation real Hilbert space and a faithful real *-representation of A. Hence, A is isometrically *-isomorphic onto a real C^*-algebra acting on a real Hilbert space.

In the rest of the if-part proof, we must show that a real C^*-algebra A acting on a real Hilbert space $\mathfrak{H}$ is isometrically *-isomorphic onto a suitable real C^*-algebra A' acting on a suitable complex Hilbert space $\mathfrak{H}_C$. Construct $\mathfrak{H}_C$ as the set of formal elements $x + iy$, where x, $y \in \mathfrak{H}$. Introduce into $\mathfrak{H}_C$ an equality relation: $x_1 + iy_1 = x_2 + iy_2$ iff $x_1 = x_2$ and $y_1 = y_2$ $(x_1, x_2, y_1, y_2 \in \mathfrak{H})$, an addition: $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$ $(x_1, x_2, y_1, y_2 \in \mathfrak{H})$, a scalar multiplication: $(\alpha + i\beta)(x + iy) = \alpha x - \beta y + i(\beta x + \alpha y)$ $(\alpha, \beta \in R, x, y \in \mathfrak{H})$, and an inner product:

$$(x_1 + iy_1, x_2 + iy_2) = (x_1, x_2) + (y_1, y_2) + i((y_1, x_2) - (x_1, y_2)) \qquad (x_1, x_2, y_1, y_2 \in \mathfrak{H}).$$

Then, $\mathfrak{H}_C$ becomes a complex Hilbert space. For each a in A, we define a mapping $a' : x + iy \mapsto ax + iay$ $(x, y \in \mathfrak{H})$. It is easy to see that a' is a bounded linear operator acting on $\mathfrak{H}_C$ with $\|a'\| = \|a\|$. Put $A' = \{a';\, a \in A\}$. The mapping: $a \mapsto a'$ gives an isometric *-isomorphism of A onto A'. This completes the if-part proof of the theorem. And the only-if-part proof of the theorem goes as usual fashion.

References

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