A REAL ANALOGUE OF THE GELFAND-NEUMARK THEOREM

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Abstract. Let $A$ be a real Banach *-algebra enjoying the following three conditions: $\|x^* x\| = \|x^*\| \|x\|$, $Sp x^* x \geq 0$, and $\|x^*\| = \|x\|$ ($x \in A$). It is shown, after Ingelstam, Palmer, and Behncke, as a real analogue of the Gelfand-Neumark theorem, that $A$ is isometrically *-isomorphic onto a real $C^*$-algebra acting on a suitable real (or complex) Hilbert space. The converse is obvious.

The aim of this note is, as a real analogue of the Gelfand-Neumark theorem, to prove the following

Theorem. A real Banach *-algebra $A$ is isometrically *-isomorphic onto a real $C^*$-algebra acting on a real (or complex) Hilbert space if and only if it satisfies the following three conditions:

1. $\|x^* x\| = \|x^*\| \|x\|$,  
2. $Sp x^* x \geq 0$, and  
3. $\|x^*\| = \|x\|$ ($x \in A$).

The above theorem was conjectured explicitly by Rickart [5, p. 181] and proved by Ingelstam [2] (cf. also Palmer [3, 4] and Behncke [1]). Their proofs were based on complexification of a real Banach *-algebra. An alternative proof which we shall give in this note will be done by real *-representation on real Hilbert space and by complexification of a real Hilbert space.

Let $A$ be a real Banach *-algebra satisfying the conditions stated in the theorem, and $H$ the set of hermitian elements in $A$. Let $R$ be the field of real numbers. In view of (2), the involution is hermitian. Put $\mu(h) = \sup(\lambda; \lambda \in \text{spec}(h))$ for all $h$ in $H$. In view of (2), $A$ is symmetric. In view of (3), the involution is continuous. So, we can make use of Rickart [5, Lemma 4.7.10] to get the sublinearity of $\mu$ on $H$, that is,

(i) $\mu(\alpha h) = \alpha \mu(h)$ and
(ii) $\mu(h + k) \leq \mu(h) + \mu(k)$,
where $0 \leq \alpha \in \mathbb{R}$, $h$, $k \in H$. Owing to the extension theorem of Hahn and Banach, for a fixed element $a$ in $A$, there exists a real linear functional, say $g$, on $H$ such that $g(h) \leq \mu(h)$ ($h \in H$) and such that $g((aa^*)^2) = \mu((aa^*)^2)$. Decompose $x = h + k$, where $h = (1/2)(x + x^*) \in H$ and $k = (1/2)(x - x^*)$ being skew adjoint. Put $f(x) = g(h)$ for all $x$ in $A$. Since $\mu(-x^*x) \leq 0$, we have $f(x^*x) \geq 0$. Thus, $f$ is a real state on $A$. It is easy to construct a *-representation real Hilbert space and a real *-representation $\psi$ of $A$. Moreover, if $aa^* \neq 0$, $\psi(a) \neq 0$. Hence, \{a; aa^* = 0\} is the *-radical of $A$, that is, the intersection of kernels of all real *-representations of $A$. In view of (1), the *-radical must be \{0\}. Thus, there exist a *-representation real Hilbert space and a faithful real *-representation of $A$. Hence, $A$ is isometrically *-isomorphic onto a real $C^*$-algebra acting on a real Hilbert space.

In the rest of the if-part proof, we must show that a real $C^*$-algebra $A$ acting on a real Hilbert space $\mathfrak{H}$ is isometrically *-isomorphic onto a suitable real $C^*$-algebra $A'$ acting on a suitable complex Hilbert space $\mathfrak{H}_c$. Construct $\mathfrak{H}_c$ as the set of formal elements $x + iy$, where $x$, $y \in \mathfrak{H}$. Introduce into $\mathfrak{H}_c$ an equality relation: $x_1 + iy_1 = x_2 + iy_2$ iff $x_1 = x_2$ and $y_1 = y_2$ ($x_1$, $x_2$, $y_1$, $y_2 \in \mathfrak{H}$), an addition: $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$ ($x_1$, $x_2$, $y_1$, $y_2 \in \mathfrak{H}$), a scalar multiplication: $(a + ib)(x + iy) = ax - by + i(bx + ay)$ ($a$, $b \in \mathbb{R}$, $x$, $y \in \mathfrak{H}$), and an inner product:

$$(x_1 + iy_1, x_2 + iy_2) = (x_1, x_2) + (y_1, y_2) + i((y_1, x_2) - (x_1, y_2))$$

Then, $\mathfrak{H}_c$ becomes a complex Hilbert space. For each $a$ in $A$, we define a mapping $a': x + iy \rightarrow ax + iay$ ($x$, $y \in \mathfrak{H}$). It is easy to see that $a'$ is a bounded linear operator acting on $\mathfrak{H}_c$ with $\|a'\| = \|a\|$. Put $A' = \{a'; a \in A\}$. The mapping: $a \rightarrow a'$ gives an isometric *-isomorphism of $A$ onto $A'$. This completes the if-part proof of the theorem. And the only-if-part proof of the theorem goes as usual fashion.

References


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