COMMON EIGENVECTORS FOR COMMUTATIVE POSITIVE LINEAR OPERATORS[1]

ROBERT E. HUFF[2]

ABSTRACT. The purpose of this note is to point out an extension of the Markov-Kakutani fixed-point theorem to a result on the existence of a common eigenvector in a cone with a compact base when acted upon by a commutative family of operators. As an application, an extension is given of a result of Kreĭn and Rutman on characteristic functionals.

Our terminology follows [5] and [7]. Let K be a compact convex subset of a locally convex Hausdorff space, and let $A(K)$ denote the space of all real-valued, continuous, affine maps on K. It is known (see e.g. [1]) that the map $g:K\rightarrow(A(K)^*$, weak*), where $q(k)(f)=f(k)$ ($f\in A(K)$, $k\in K$), is an affine homeomorphism of K onto the set of all positive linear functionals on $A(K)$ of norm one. Hence, $q(K)$ is a compact base for the cone P of positive linear functionals in ($A(K)^*$, weak*). Every continuous affine map σ on K to K gives rise to a unique continuous, order-preserving, linear operator σ' on $A(K)^*$ such that $\sigma'(q(K))\subset q(K)$ and $\sigma=q^{-1}\sigma' q$. If Σ is a commutative semigroup of continuous affine maps on K to K, then the Markov-Kakutani Theorem (cf. [4, p. 456]) asserts the existence of a common fixed-point in K; equivalently, the existence of a common eigenvector, corresponding to the common eigenvalue 1, in P for the semigroup $\Sigma'=\{\sigma':\sigma\in\Sigma\}$. We shall prove the following result which, by the above remarks, includes the Markov-Kakutani Theorem.

THEOREM 1. *Let E be an ordered, locally convex Hausdorff space whose positive cone P has a compact base B. Let Σ be a commutative semigroup of continuous, order-preserving linear operators on E. Then there exists a point b_0 in B such that for every σ in Σ*

$$\sigma(b_0) = \lambda_\sigma b_0$$

Received by the editors July 8, 1969.

AMS Subject Classifications. Primary 4620; Secondary 4610, 4660, 1530.

Key Words and Phrases. Common eigenvectors, fixed-points, locally convex spaces, ordered linear spaces, Markov-Kakutani Theorem, compact base, characteristic functional.

[1] This research is part of the author's doctoral dissertation prepared at the University of North Carolina under the direction of Professor B. J. Pettis, whose guidance is gratefully acknowledged.

[2] This research was done while the author was supported by a National Science Foundation Fellowship.

for some $\lambda_\sigma \geqq 0$; *i.e.*, b_0 *is a common eigenvector for the members of* Σ.

Note that since B is a base for P, b_0 in the theorem is necessarily nonzero [7, p. 25].

The Schauder-Tychonoff fixed-point theorem will be used at an essential point in the proof.

Day [3] extended the Markov-Kakutani Theorem by replacing "commutative semigroup" by "amenable semigroup" in the hypotheses. That this is not possible in Theorem 1 is demonstrated by the following example (from [8, p. 472]): consider the plane E^2 with cone $P=\{(x, y): x\geqq 0, y\geqq 0\}$ which has compact base $B=\{(x, y)\in P: x+y=1\}$. Let Σ be the multiplicative group generated by all diagonal 2×2 matrices with positive diagonal entries, together with the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Σ is a solvable group, and hence amenable (cf. [2]), but there is no common eigenvector in B under the action of Σ.

PROOF OF THEOREM 1. Since B is a base for P, for every element y in P there is a unique real number $\alpha(y)\geqq 0$ such that $y=\alpha(y)\beta(y)$ for some $\beta(y)$ in B; if y is nonzero, then $\beta(y)$ is unique. We first show that $\alpha: y\rightarrow\alpha(y)$ $(y\in P)$ and $\beta: y\rightarrow\beta(y)$ $(y\in P,\ y\neq 0)$ define continuous functions on the respective domains.

(1) Suppose k is a positive real number, suppose $\{y_\lambda\}_\lambda$ is a net in $[0, k]B=\{\delta b: \delta\in[0, k], b\in B\}$, and suppose $\{y_\lambda\}_\lambda$ converges to some y in P. By compactness of $[0, k]$ and B, and by the Hausdorff property of E, every subnet of $\{\alpha(y_\lambda)\}_\lambda$ has a sub-subnet which converges to $\alpha(y)$. Hence, $\{\alpha(y_\lambda)\}_\lambda$ converges to $\alpha(y)$; and so α is continuous on $[0, k]B$.

Now let y be any point in P. There is some real number $k>1$ such that y is in $[0, k-1]B$. By the separation theorem (cf. [5, p. 119]) applied to the disjoint compact convex sets $[0, k-1]B$ and kB, there is a continuous linear functional f on E and a real number γ such that

$$\sup f([0, k-1]B) < \gamma < \inf f(kB).$$

Then $U=\{x\in P: f(x)<\gamma\}$ is a neighborhood of y in P and is contained in $[0, k]B$. Since α is continuous on $[0, k]B$, it is continuous on U and hence at y.

(2) Suppose $\{y_\lambda\}_\lambda$ is a net in P with $y_\lambda\neq 0$ (all λ), and suppose $\{y_\lambda\}_\lambda$ converges to $y\neq 0$ in P. By (1), $\{\alpha(y_\lambda)\}_\lambda$ converges to $\alpha(y)$. By compactness of B and the fact that $\alpha(y)$ is positive, every subnet of

$\{\beta(y_\lambda)\}_\lambda$ has a sub-subnet which converges to $\beta(y)$. Hence, $\{\beta(y_\lambda)\}_\lambda$ converges to $\beta(y)$; and so β is continuous.

Now let $\mathcal{K}$ denote the collection of all nonvoid, compact, convex subsets K of B satisfying

$$\sigma(K) \subset [0, \infty)K, \quad \text{all } \sigma \text{ in } \Sigma.$$

If $\mathcal{K}$ is ordered by inclusion, compactness and Zorn's Lemma imply there is some minimal member K_0 of $\mathcal{K}$.

Choose any σ in Σ. We argue that there is some real number $\lambda_\sigma \geqq 0$ such that the set

$$K(\lambda_\sigma) = \{k \in K_0 : \sigma(k) = \lambda_\sigma k\}$$

is nonvoid. If $0 \in \sigma(K_0)$, choose $\lambda_\sigma = 0$. If $0 \notin \sigma(K_0)$, then $\beta \circ \sigma | K_0$ is a continuous map on K_0 to K_0. By the Schauder-Tychonoff Theorem (cf. [4, p. 456]), there exists some k_0 in K such that $\beta \circ \sigma(k_0) = k_0$. Then $\sigma(k_0) = \alpha(\sigma(k_0))k_0$, so that we may choose $\lambda = \alpha(\sigma(x_0))$.

Clearly $K(\lambda_\sigma)$ is compact and convex. Suppose k is in $K(\lambda_\sigma)$. Choose any τ in Σ. By commutativity

$$\sigma(\tau(k)) = \tau(\sigma(k)) = \tau(\lambda_\sigma k) = \lambda_\sigma(\tau(k)).$$

This implies that $\tau(k)$ is in $[0, \infty)K(\lambda_\sigma)$. Since k was arbitrary in $K(\lambda_\sigma)$, $\tau(K(\lambda_\sigma)) \subset [0, \infty)K(\lambda_\sigma)$. Since τ was arbitrary in Σ, $K(\lambda_\sigma)$ is in $\mathcal{K}$. By minimality of K_0, $K(\lambda_\sigma) = K_0$. Since σ was arbitrary in Σ, one concludes that for every σ in Σ there is some $\lambda_\sigma \geqq 0$ such that every element k in K_0 satisfies $\sigma(k) = \lambda_\sigma k$. Q.E.D.

As an application of Theorem 1, we extend the following result:

THEOREM (KREĬN-RUTMAN [6, 3.3]). *Suppose E is an ordered, normed linear space whose positive cone P is closed and has nonvoid topological interior P^i. Let Σ be a commutative semigroup of linear operators on E such that $\sigma(P^i) \subset P^i$* for *each σ in Σ. Then there exists a nontrivial, continuous, positive linear functional f_0 on E such that for every σ in Σ*

$$f_0 \circ \sigma = \lambda_\sigma f_0$$

for some scalar $\lambda_\sigma > 0$.

If a linear space E is ordered by a cone P, then P^0 will denote the set of all order units in P [7, p. 4]. P^0 is the "radial kernel" of P in the terminology of [5, p. 14] (the set of all "internal points" of P in the terminology of [4, p. 410]). P^0 equals the topological interior P^i of P whenever E is a linear topological space and P^i is nonvoid (cf. [4, p.

413]). If P^0 is nonvoid, every nonzero positive linear functional on E is strictly positive at every member of P^0.

THEOREM 2. *Suppose E is an ordered linear space such that the positive cone P has nonvoid radial kernel P^0. Let Σ be a commutative semigroup of order-preserving, linear operators on E. Then there exists a nontrivial positive linear functional f_0 on E such that for every σ in Σ*

$$f_0 \circ \sigma = \lambda_\sigma f_0$$

for some $\lambda_\sigma \geqq 0$.

If σ is in Σ and if $\sigma(P^0) \cap P^0$ is nonvoid, then $\lambda_\sigma > 0$.

If σ is in Σ and if $\sigma(y_0) = \mu_\sigma(y_0)$ for some y_0 in P^0 and some scalar μ_σ, then $\lambda_\sigma = \mu_\sigma$. In particular, if for every σ in Σ there is some σ-fixed-point in P^0, then f_0 is a common fixed-point for the adjoints of the members of Σ.

As an example of a situation satisfying the hypotheses of Theorem 2, but not those in the Kreĭn-Rutman Theorem, take the nonnegative, diagonal 2×2 matrices acting on the plane ordered by the cone $\{(x, y): x \geqq 0, y \geqq 0\}$.

PROOF OF THEOREM 2. Let E' denote the algebraic dual of E, let E' have the topology of pointwise convergence on E, and let P' denote the cone of all positive linear functionals on E. P' is nontrivial (cf. [5, p. 23, (3.2)]). The adjoint σ' of a member σ of Σ is continuous, linear, and maps P' into P'. Let p_0 be any point in the radial kernel of P, and let $B' = \{f \in P': f(p_0) = 1\}$. Then B' is a base P'. Furthermore, since for every y in E there exists a real number $\delta_y > 0$ such that

$$-\delta_y p_0 \leqq y \leqq \delta_y p_0$$

in E, B' is homeomorphic to a closed subset of the product space $\prod\{[-\delta_y, \delta_y]: y \in E\}$. Thus, B' is a compact base for P'. By Theorem 1, there exists an element f_0 in B' such that for every σ in Σ

$$f_0 \circ \sigma = \sigma'(f_0) = \lambda_\sigma f_0$$

for some $\lambda_\sigma \geqq 0$.

Suppose σ is in Σ and y_0 is in $\sigma(P^0) \cap P^0$. Then $f_0 \circ \sigma(y_0) > 0$ and $f_0(y_0) > 0$, so that $\lambda_\sigma > 0$.

Suppose σ is in Σ and the y_0 is some point in P^0 such that $\sigma(y_0) = \mu_\sigma y_0$ for some scalar μ_σ. Then

$$\lambda_\sigma f_0(y_0) = f_0 \circ \sigma(y_0) = f_0(\mu_\sigma y_0) = \mu_\sigma f_0(y_0).$$

Since $f_0(y_0) > 0$, $\lambda_\sigma = \mu_\sigma$. Q.E.D.

Finally, we remark that, following Silverman and Yen's modification of the Kreĭn-Rutman result [8], one can weaken slightly the assumption of commutativity in both Theorem 1 and Theorem 2. If one replaces commutativity of Σ in Theorem 1 by the following assumption:

(A1) there is a subset S of Σ such that

(1) there is some point $\bar{b}$ in B satisfying $s(\bar{b})=\bar{b}$ for every s in S, and

(2) for every pair σ_1, σ_2, in Σ there are elements s_1, s_2 in S such that $\sigma_1\sigma_2 s_1=\sigma_2\sigma_1 s_2$,

then the set $B_1=\{b\in B: s(b)=b \text{ for every } s \text{ in } S\}$ is a compact base for the cone $P_1=[0, \infty)B_1$ which it generates. Using (A1)(2), it can be shown that $\sigma(P_1)\subset P_1$ for every σ in Σ, and that the elements of Σ commute on the linear span of P_1. Hence, *the conclusion of* Theorem 1 *remains valid if the assumption of commutativity of* Σ *is replaced by assumption* (A1). Using this extended result in the proof of Theorem 2, we have that *the conclusions of* Theorem 2 *remain valid if the assumption of commutativity of* Σ *is replaced by the following assumption*:

(A2) there is a subset S of Σ such that

(1) there is some nonzero f_1 in P' satisfying $f_1\circ s=f_1$ for all s in S, and

(2) for every pair σ_1, σ_2 in Σ there are elements s_1, s_2 in S such that $s_1\sigma_1\sigma_2=s_2\sigma_2\sigma_1$.

References

1. L. N. Argabright, *Invariant means and fixed points: A sequel to Mitchell's paper*, Trans. Amer. Math. Soc. **130** (1968), 127–130. MR **36** #667.

2. M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544. MR **19**, 1067.

3. ———, *Fixed point theorems for compact convex sets*, Illinois J. Math. **5** (1961), 585–590. MR **25** #1547.

4. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

5. J. L. Kelley, I. Namioka, et al, *Linear topological spaces*, The University Series in Higher Math., Van Nostrand, Princeton, N. J., 1963. MR **29** #3851.

6. M. G. Kreĭn and M. R. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Uspehi Mat. Nauk **3** (1948), no. 1 (23), 3–95; English transl., Amer. Math. Soc. Transl. (1) **10** (1962), 199–325. MR **10**, 256.

7. A. L. Peressini, *Ordered topological vector spaces*, Harper & Row, New York, 1967. MR **37** #3315.

8. R. J. Silverman and T. Yen, *Characteristic functionals*, Proc. Amer. Math. Soc. **10** (1959), 471–477. MR **22** #187.

University of North Carolina, Chapel Hill, North Carolina 27514