COMPLETE CONTINUITY OF THE INVERSE OF A POSITIVE SYMMETRIC OPERATOR[1]

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ABSTRACT. Let A be a symmetric positive definite linear transformation defined on a dense subset of a Hilbert space H, and let H_A be the Hilbert space completion of the domain of A with respect to the inner product $(u, v)_A = (Au, v)$. It is shown that the inverse of A is completely continuous on H_A if and only if it is completely continuous on H.

I. **Introduction.** In certain problems in mathematical physics, one often encounters differential operators which are defined on a dense subset of a Hilbert space and, under appropriate boundary conditions, are symmetric and semibounded. However, differential operators fail to possess certain desirable properties usually enjoyed by their inverses. One of these properties is complete continuity, and in this paper we show the equivalence of the complete continuity of this inverse in two Hilbert spaces (Theorem 2).

II. **Preliminaries.** Let A be a symmetric linear transformation defined on a linear subset M which is dense in a Hilbert space H. Suppose also that A is positive definite on M; that is, suppose there exists a constant $\gamma > 0$ such that

$$(Au, u) \geqq \gamma^2 \|u\|^2, \qquad u \in M.$$

Following the method due to K. Friedrichs [1] (cf. also [2], [3]), we define a new scalar product $(u, v)_A$ on M by setting

$$(u, v)_A = (Au, v), \qquad u, v \in M.$$

We denote the corresponding norm by

$$\|u\|_A = (Au, u)^{1/2}, \qquad u \in M.$$

With this new metric, M becomes an inner product space which can

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be completed in the usual way, thereby obtaining a Hilbert space H_A.

In the case we are considering, we have the following theorem:

THEOREM 1. *The Hilbert space H_A can be identified with a subspace of H.*

For a proof of this theorem, see [1] and also [2], [3].

It is easy to show that

$$(u, v)_A = (Au, v), \qquad u \in M, \qquad v \in H_A,$$
$$\|u\|_A \geqq \gamma\|u\|, \qquad u \in H_A.$$

Let f be a fixed element in H and consider the linear functional

$$F_f(u) = (u, f), \qquad u \in H_A.$$

F_f is a bounded linear functional on H_A, and hence by the Riesz Representation Theorem there exists a unique element $u_f \in H_A$ such that

$$F_f(u) = (u, f) = (u, u_f)_A, \qquad u \in H_A.$$

We thus define an operator G from H into H_A by setting

$$Gf = u_f, \qquad f \in H.$$

We then have that

$$(u, f) = (u, Gf)_A, \qquad u \in H_A, \qquad f \in H.$$

The properties of the operator G are summarized in the following two propositions, the proofs of which can be extracted from [2] and [3].

PROPOSITION 1. *G is a positive symmetric bounded linear transformation both on H and on H_A.*

PROPOSITION 2. *If $Gf = 0$ for $f \in H$, then $f = 0$. In other words, G has an inverse.*

III. **Complete continuity of G.** In the applications, it is useful to know whether or not the operator G is completely continuous (compact) as an operator on H_A. This question is not always as simple to answer as the related question of the complete continuity of G as an operator on H. The next theorem provides the desired connection.

THEOREM 2. *G is completely continuous on H_A if and only if G is completely continuous on H.*

PROOF. For the sake of clarity, we use the notation $\sum$ to denote summation in the H metric of an infinite series and introduce the notation $\sum_A$ to denote summation in the H_A metric.

Suppose first that G is completely continuous on H. Since G is also symmetric on H, its nonzero eigenvalues μ_i, arranged according to decreasing absolute values, are of finite multiplicity and either finite or countably infinite in number (cf. [2]). In the latter case, $\mu_i \to 0$. Furthermore, every element of the form Gf, $f \in H$, can be developed in terms of the orthonormal system $\{\phi_i\}$ of corresponding eigenvectors:

$$Gf = \sum (Gf, \phi_i)\phi_i = \sum \mu_i(f, \phi_i)\phi_i.$$

Since G is a positive operator on H, $\mu_i > 0$. By Proposition 2, 0 is not an eigenvalue of G and consequently the eigenvectors ϕ_i corresponding to the nonzero eigenvalues μ_i form a complete orthonormal sequence in H (cf. [2]). Since

$$G\phi_i = \mu_i\phi_i, \qquad \mu_i > 0,$$

we have that

$$\phi_i = \mu_i^{-1} G\phi_i \in H_A.$$

Setting $u_i = \mu_i^{1/2}\phi_i$, we observe that the u_i form an orthonormal set in H_A.

Let $u \in H_A$ and set

$$v = \sum_A (u, u_i)_A u_i.$$

Since G is a continuous linear transformation on H_A, we have that

$$\begin{aligned} Gv &= \sum_A (u, u_i)_A G u_i \\ &= \sum_A \mu_i (u, u_i)_A u_i \\ &= \sum \mu_i (u, u_i)_A u_i. \end{aligned}$$

The last step follows because convergence in the H_A metric implies convergence in the H metric.

Now G is completely continuous and symmetric on H so that

$$\begin{aligned} Gu &= \sum \mu_i (u, \phi_i)\phi_i \\ &= \sum (u, u_i) u_i \\ &= \sum (u, Gu_i)_A u_i \\ &= \sum \mu_i (u, u_i)_A u_i. \end{aligned}$$

Consequently, $Gu = Gv$ and $u = v$. Therefore, the u_i form a complete orthonormal set in H_A and

$$Gu = \sum_A \mu_i(u, u_i)_A u_i, \qquad u \in H_A.$$

It follows that G is completely continuous on H_A (cf. [2]).

Conversely, suppose that G is completely continuous on H_A. Since G is symmetric on H_A, its nonzero eigenvalues μ_i, arranged according to decreasing absolute values, are of finite multiplicity and either finite or countably infinite in number. In the latter case, $\mu_i \to 0$.

Since G is a positive operator on H_A, $\mu_i > 0$. By Proposition 2, 0 is not an eigenvalue of G and consequently the eigenvectors u_i corresponding to the nonzero eigenvalues μ_i form a complete orthonormal sequence in H_A. Setting $\phi_i = \mu_i^{-1/2} u_i$, we see that the ϕ_i form an orthonormal set in H.

Let $f \in H$. Then $Gf \in H_A$ and

$$Gf = \sum_A (Gf, u_i)_A u_i.$$

Since convergence in H_A implies convergence in H, we obtain

$$\begin{aligned} Gf &= \sum (Gf, u_i)_A u_i \\ &= \sum (f, u_i) u_i \\ &= \sum \mu_i (f, \phi_i) \phi_i \qquad f \in H. \end{aligned}$$

Hence, G is completely continuous on H.

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