A NONEMBEDDING THEOREM FOR FINITE GROUPS

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ABSTRACT. Let N be the class of nilpotent groups with the following properties:

(1) The center of N, $Z_1(N)$ is of prime order.

(2) There exists an abelian characteristic subgroup A of N such that $Z_1(N) \subset A \subseteq Z_2(N)$ where $Z_2(N)$ is the second term in the upper central series of N.

The main result shown is the following: If $N \in \mathfrak{X}$, then N cannot be an invariant subgroup contained in the Frattini subgroup of a finite group.

Hobby has shown in [2] that a nonabelian group whose center is cyclic can not be the Frattini subgroup of any p-group. Chao in [1] has shown that a nonabelian group whose center is of prime order cannot be embedded in the derived group of any nilpotent group. The result obtained here is of a similar nature. All groups considered here are finite.

DEFINITION. Let $\mathfrak{X}$ be the class of nilpotent groups N which have the following properties:

(1) The center of N, $Z_1(N)$, is of prime order.

(2) There exists an abelian characteristic subgroup A of N such that $Z_1(N) \subset A \subseteq Z_2(N)$ where $Z_2(N)$ is the second term in the upper central series of N.

If N is an arbitrary group and N' is the derived group of N, then $N' \cap Z_2(N)$ is an abelian characteristic subgroup of N. If N is nilpotent and $N' \cap Z_2(N) \subseteq Z_1(N)$, then $N' \subseteq Z_1(N)$ and N has nilpotent length 1 or 2. Hence if N has nilpotent length greater than 2 and $Z_1(N)$ is of prime order, then $N \in \mathfrak{X}$.

The main result shown here is the following

THEOREM. *If $N \in \mathfrak{X}$, then N can not be an invariant subgroup contained in the Frattini subgroup of any group.*

Clearly if the center of N is of prime order, then each term in the upper central series of N is a p-group. Futhermore $Z_2(N)/Z_1(N)$ is elementary abelian. If $N \in \mathfrak{X}$ and A is as in the definition, then $\bar{A} = A/Z_1(N)$ is elementary abelian and is therefore the direct product of cyclic groups of order p, denoted by ${}_1C_p, \cdots, {}_kC_p$. In the

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natural way, $\bar{A}$ will be considered as a k-dimensional vector space over F_p, the field of p elements. We denote by η the natural mapping from A onto $\bar{A}$ and by $\mathrm{Inn}_A(N)$ the group of automorphisms of A induced by elements of N. If N is invariant in G, then it will be shown that $\mathrm{Inn}_A(N)$ has a complement in $\mathrm{Inn}_A(G)$.

For notational convenience let $Z=Z_1(N)$. Let $S=\{\sigma\in\mathrm{Aut}(A);\ \sigma(a)a^{-1}\in Z$ for all $a\in A$, $\sigma(z)=z$ for all $z\in Z\}$. S is a subgroup of $\mathrm{Aut}(A)$ and $\mathrm{Inn}_A(N)$ is a subgroup of S.

LEMMA 1. *The set S with operations of composition and scaler multiplication defined by $r\rho=\rho^r$ for $r\in F_p$ and $\rho\in S$ is a vector space over F_p and* $\dim S=\dim \bar{A}$. $\mathrm{Inn}_A(N)$ *is a subspace of* S.

PROOF. For $\sigma\in S$, let f_σ be the mapping from $\bar{A}$ into Z defined by $f_\sigma(\bar{a})=\sigma(a)a^{-1}$ where $a\in A$ such that $\eta(a)=\bar{a}$. It is easily verified that $f_\sigma(\bar{a})$ is independent of the choice of a and that $f_\sigma\in\mathrm{Hom}_{F_p}(\bar{A}, Z)$. For $f\in\mathrm{Hom}_{F_p}(\bar{A}, Z)$, let σ_f be the mapping from A into A defined by $\sigma_f(a)=f(\bar{a})a$ where $\bar{a}=\eta(a)$. One verifies that $\sigma_f\in S$. Define θ to be the mapping from S into $\mathrm{Hom}_{F_p}(\bar{A}, Z)$ defined by $\theta(\sigma)=f_\sigma$ and τ to be the mapping from $\mathrm{Hom}_{F_p}(\bar{A}, Z)$ into S defined by $\tau(f)=\sigma_f$. Then θ is a vector space isomorphism with inverse τ. Therefore $S\cong\mathrm{Hom}_{F_p}(\bar{A}, Z)=\bar{A}^*\cong\bar{A}$, where $\bar{A}^*$ is the dual of $\bar{A}$. Clearly $\mathrm{Inn}_A(N)$ is a subspace of S.

LEMMA 2. $\mathrm{Inn}_A(N)=S$.

PROOF. For $n\in N$, let σ_n be the automorphism of A induced by n. If $\bar{a}\in A$ is annihilated by $\theta(\mathrm{Inn}_A(N))$, then $f_{\sigma_n}(\bar{a})=1$ for all $n\in N$. Let $a\in A$ such that $\eta(a)=\bar{a}$. Then $nan^{-1}a^{-1}=1$ for all $n\in N$ and therefore $a\in Z$. Thus $\bar{a}$ is the identity of $\bar{A}$ and $\mathrm{Inn}_A(N)=S$.

Let z be a generator of Z, $\bar{x_i}$ be a generator of ${}_iC_p$ and $x_i\in A$ such that $\eta(x_i)=\bar{x_i}$ for $i=1,\cdots,k$. The ordering of the indices in the decomposition of $\bar{A}$, the choice of $\bar{x_i}$ and x_i will be considered as fixed. The representation of $a=x_1^{a_1}\cdots x_k^{a_k}z^s$, $a_1,\cdots,a_k$, $s\in F_p$ is easily seen to be unique.

We now find a particular basis for $S=\mathrm{Inn}_A(N)$. Define, for $j=1,\cdots,k$, the mapping e_j from A into A by

$$e_j(x_1^{a_1}\cdots x_k^{a_k}z^s)=x_1^{a_1}\cdots x_k^{a_k}z^{(s+a_j)}.$$

Since each $a\in A$ is uniquely expressible in the form indicated, each e_j is well defined. One can then show that the set $e_1,\cdots,e_k$ is a basis for S.

LEMMA 3. *Let $N\in\mathfrak{X}$ and N be invariant in G. Then* $\mathrm{Inn}_A(N)$ *is complemented in* $\mathrm{Inn}_A(G)$.

PROOF. Let $M=\{B\in \mathrm{Inn}_A(G);\ B(x_i)=x_1^{a_{1i}}\cdots x_k^{a_{ki}}$ for $i=1,\cdots,k$ where $a_{1i},\cdots,a_{ki}\in F_p\}$. Since A is abelian, M is a subgroup of $\mathrm{Inn}_A(G)$. Now let $R\in \mathrm{Inn}_A(G)$. Then $R(x_i)=x_1^{a_{1i}}\cdots x_k^{a_{ki}}z^{s_i}$ for $i=1,\cdots,k$ and $R(z)=z^s$. Let $t_i\in F_p$ such that $s_i+t_is=0$ for $i=1,\cdots,k$. Then $Re_1^{t_1}\cdots e_k^{t_k}e_1^{-t_1}\cdots e_k^{-t_k}=R$ and, using Lemma 2, $Re^{t_1}\cdots e_k^{t_k}\in M$ and $e^{-t_1}\cdots e^{-t_k}\in S$. Hence $\mathrm{Inn}_A(G)=\mathrm{Inn}_A(N)\cdot M$ and since $M\cap \mathrm{Inn}_A(N)$ clearly equals the identity automorphism of A, M complements $\mathrm{Inn}_A(N)$ in $\mathrm{Inn}_A(G)$.

PROOF OF THEOREM. Suppose $N\in\mathfrak{X}$ and that G is a group such that N is invariant in G and $N\subseteq\phi(G)$ where $\phi(G)$ denotes the Frattini subgroup of G. Let f be the homomorphism which assigns to each element of G the automorphism which it induces in A. Then

$$\mathrm{Inn}_A(N)=f(N)\subseteq f(\phi(G))\subseteq\phi(f(G))=\phi(\mathrm{Inn}_A(G)).$$

However, by Lemma 3, $\mathrm{Inn}_A(N)$ is complemented in $\mathrm{Inn}_A(G)$. This contradiction establishes the result.

COROLLARY. *A nilpotent group of length greater than 2 whose center is of prime order cannot be an invariant subgroup contained in the Frattini subgroup of any group.*

REFERENCES

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