

METRIC DEFINITION OF THE LINEAR STRUCTURE

RUSSELL G. BILYEU

Mazur and Ulam [1] proved that the metric of a real normed linear space determines the linear operations. The following converse of an elementary theorem gives an explicit characterization of convex combinations in terms of the metric d of such a space.

THEOREM. *If $0 < a < 1$ and if x , y , and z are members of a real normed linear space such that for each w in the space*

$$d(w, z) \leq ad(w, x) + (1 - a)d(w, y)$$

then $z = ax + (1 - a)y$.

PROOF. We may assume $z = 0$ and $a \leq 1/2$. Taking $a = 1/2$, suppose $2d(w, 0) \leq d(w, x) + d(w, y)$ for all w . Then 0 is in the set T of those nonnegative integers t for which, for each positive integer m ,

$$(2m + t)d(x + y, 0) \leq d(mx + my, x) + d(mx + my, y).$$

Choosing t in T , observe that if m is a positive integer then so is $2m$, and

$$(4m + t)d(x + y, 0) \leq d(2mx + 2my, x) + d(2mx + 2my, y).$$

Setting $w = (2m - 1)x + 2my$ in the hypothesis leads to

$$d(2mx + 2my, x) \leq d(mx + my, x) + (m - 1/2)d(x + y, 0).$$

This last statement also holds with x and y interchanged, and from these inequalities we conclude that

$$(4m + t)d(x + y, 0) \leq d(mx + my, x) + d(mx + my, y) \\ + (2m - 1)d(x + y, 0).$$

Therefore $(2m + t + 1)d(x + y, 0) \leq d(mx + my, x) + d(mx + my, y)$. By induction, T contains all nonnegative integers. In particular, if t is a nonnegative integer then

$$(2 + t)d(x + y, 0) \leq d(y, 0) + d(x, 0).$$

This implies $x + y = 0$, as desired.

Now if $0 < a < 1/2$ and if

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$$d(w, 0) \leq ad(w, x) + (1 - a)d(w, y),$$

then

$$\begin{aligned}d(w, 0) &\leq d(aw, ax) + d(w - aw, y - ay) \\ &\leq d(aw, ax) + d(aw, y - ay) + d(w - 2aw, 0) \\ &= d(aw, ax) + d(aw, y - ay) + (1 - 2a)d(w, 0).\end{aligned}$$

Therefore $2d(aw, 0) \leq d(aw, ax) + d(aw, y - ay)$. If these inequalities are assumed to hold for all w , then by the previous argument, $ax + y - ay = 0$.

BIBLIOGRAPHY

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NORTH TEXAS STATE UNIVERSITY, DENTON, TEXAS 76203