METRIC DEFINITION OF THE LINEAR STRUCTURE

RUSSELL G. BILYEU

Mazur and Ulam [1] proved that the metric of a real normed linear space determines the linear operations. The following converse of an elementary theorem gives an explicit characterization of convex combinations in terms of the metric d of such a space.

THEOREM. *If* $0<a<1$ *and if* x, y, *and* z *are members of a real normed linear space such that for each* w *in the space*

$$d(w, z) \leqq ad(w, x) + (1 - a)d(w, y)$$

then $z=ax+(1-a)y$.

PROOF. We may assume $z=0$ and $a \leqq 1/2$. Taking $a=1/2$, suppose $2d(w, 0) \leqq d(w, x)+d(w, y)$ for all w. Then 0 is in the set T of those nonnegative integers t for which, for each positive integer m,

$$(2m + t)d(x + y, 0) \leqq d(mx + my, x) + d(mx + my, y).$$

Choosing t in T, observe that if m is a positive integer then so is $2m$, and

$$(4m + t)d(x + y, 0) \leqq d(2mx + 2my, x) + d(2mx + 2my, y).$$

Setting $w=(2m-1)x+2my$ in the hypothesis leads to

$$d(2mx + 2my, x) \leqq d(mx + my, x) + (m - 1/2)d(x + y, 0).$$

This last statement also holds with x and y interchanged, and from these inequalities we conclude that

$$\begin{aligned}(4m + t)d(x + y, 0) \leqq\ & d(mx + my, x) + d(mx + my, y) \\ & + (2m - 1)d(x + y, 0).\end{aligned}$$

Therefore $(2m+t+1)d(x+y, 0) \leqq d(mx+my, x)+d(mx+my, y)$. By induction, T contains all nonnegative integers. In particular, if t is a nonnegative integer then

$$(2 + t)d(x + y, 0) \leqq d(y, 0) + d(x, 0).$$

This implies $x+y=0$, as desired.

Now if $0<a<1/2$ and if

Received by the editors September 19, 1969.

AMS Subject Classifications. Primary 4610; Secondary 5250.

Key Words and Phrases. Metric definition, linear structure, normed linear space, convex combinations.

$$d(w, 0) \leqq ad(w, x) + (1 - a)d(w, y),$$

then

$$\begin{aligned} d(w, 0) &\leqq d(aw, ax) + d(w - aw, y - ay) \\ &\leqq d(aw, ax) + d(aw, y - ay) + d(w - 2aw, 0) \\ &= d(aw, ax) + d(aw, y - ay) + (1 - 2a)d(w, 0). \end{aligned}$$

Therefore $2d(aw, 0) \leqq d(aw, ax) + d(aw, y - ay)$. If these inequalities are assumed to hold for all w, then by the previous argument, $ax + y - ay = 0$.

Bibliography

1. S. Mazur and S. Ulam, *Sur les transformations isometriques d'espaces vectoriels*, C.R. Acad. Sci. Paris **194** (1932), 946–948.

North Texas State University, Denton, Texas 76203