DECOMPOSABLE COMPACT CONVEX SETS AND PEAK SETS FOR FUNCTION SPACES[1]

LEONARD ASIMOW

ABSTRACT. Geometric conditions are known under which a closed face of a compact convex set is a peak set with respect to the space of continuous affine (real-valued) functions. The purpose of this note is to give an application of this "abstract-geometric" set-up to the problem of finding peak sets (or points) in a compact Hausdorff space with respect to a closed subspace of continuous complex-valued functions. In this fashion we obtain the *strong hull* criteria of Curtis and Figá-Talamanca and in particular the Bishop peak point theorem for function algebras.

Let X be a compact convex subset of a Hausdorff locally convex space and let $A(X)$ denote the space of continuous real-valued affine functions on X. Then $A(X)$ is a Banach space under the supremum norm. We make the usual identification of X with the set of positive normalized functionals in $A(X)^*$ with the weak* topology. If F is a closed face of X we say X is *decomposable* at F under f [2] if $f\in A(X)^{**}$ such that f is identically zero on the weak* closed linear span of F and

$$X = \operatorname{conv}(F \cup \{x \in X : f(x) \geqq 1\}).$$

We say F is an (A) *peak face* of X (within G_δ's) if for any G_δ set G in X containing F there is a nonnegative $h\in A(X)$ such that

$$F \subset \{x \in X : h(x) = 0\} \subset G.$$

Our basic tool in what follows is the fact that if X is decomposable at F then F is an (A) peak face of X (within G_δ's) [1].

Let Ω be a compact Hausdorff space and let M be a uniformly closed separating subspace of continuous complex-valued functions (including the constants). Let X be the state space of M, i.e., $X=\{x\in M^*:\|x\|=1=\langle 1, x\rangle\}$. Then X is compact and convex in the weak* topology. If ext X denotes the set of extreme points of X then Ω is homeomorphic (under the evaluation map ϕ) to $(\text{ext } X)^-$. If $x\in\Omega$ and $f\in M$ we write $f(x)$ for $\langle f, \phi x\rangle$. If $f\subset M$ then $\mathrm{Re}(f)$ is in $A(X)$. We deal with the real *and* imaginary parts of f by considering

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in place of X the compact convex set

$$Z \equiv \operatorname{conv}(X \cup -iX).$$

For each $f \in M$ let $\theta f \in A(Z)$ be defined by

$$\theta f(z) = \operatorname{Re}\langle f, z\rangle.$$

Then $\theta f(-iz) = \operatorname{Re}\langle f, -iz\rangle = \operatorname{Im}\langle f, z\rangle$.

PROPOSITION 1. *The map $\theta: M \to A(Z)$ is an isomorphism (bounded, one-to-one and onto).*

PROOF. Let $h \in M$ and choose $x \in \operatorname{ext} X$ such that $|h(x)| = \|h\| = 1$. Then either $|\operatorname{Re} h(x)| \geqq \frac{1}{2}$ or $|\operatorname{Im} h(x)| \geqq \frac{1}{2}$. Thus either $|\theta h(x)| \geqq \frac{1}{2}$ or $|\theta h(-ix)| \geqq \frac{1}{2}$. Hence

$$\|\theta h\| \leqq \|h\| \leqq 2\|\theta h\|.$$

Since the range of θ is always dense this shows θ is onto.

We shall say the closed face F of X is an (M) peak face (within G_δ's) if for each G_δ set G in X containing F there is an $f \in M$ such that $\|f\| = 1$ and

$$F \subset \{x \in X : f(x) = 1\} \subset \{x \in X : |f(x)| = 1\} \subset G.$$

THEOREM 2. *If F is a closed face of X such that Z is decomposable at $\operatorname{conv}(F \cup -iF)$ then F is an (M) peak face of X (within G_δ's).*

PROOF. If G is a G_δ of X containing F then by Proposition 1 there is an $h \in M$ such that

$$0 \leqq \theta h \leqq \sqrt{2}/2 \quad \text{on } Z, \quad \theta h \equiv 0 \quad \text{on } \operatorname{conv}(F \cup -iF),$$
$$0 < \theta h \quad \text{on } Z \backslash \operatorname{conv}(G \cup -iG).$$

Keeping in mind that $\theta h(-iz) = \operatorname{Im}\langle h, z\rangle$ these properties say that

(1) $h(X)$ is contained in the square inscribed in the first quadrant of the unit disk.

(2) $h(F) = \{0\}$,

(3) if $x \in X \backslash G$ then $\operatorname{Re}\langle h, x\rangle$, $\operatorname{Im}\langle h, x\rangle > 0$. Thus $1 - e^{-i\pi/4}h \in M$ is easily seen to be a function of the required type.

Let F be a closed subset of Ω and let $F^\perp = \{f \in M : f \equiv 0 \text{ on } F\}$. Let $N \equiv F^{\perp\perp}$ be the polar of $F^\perp$ in M^* and let $\hat{F} = N \cap X$.

Following Curtis and Figá-Talamanca [4] we say F is a *strong hull* if there is an $r > 0$ such that for any neighborhood V of F and any $\epsilon > 0$ there is an $f \in M$ such that

$$f \equiv 0 \quad \text{on } F, \quad |f(y) - 1| < \epsilon \quad \text{for } y \in \Omega \backslash V, \quad \|f\| \leqq r.$$

Let F be a strong hull. Then for any $x \in \Omega \backslash F$ we have $\phi x \in X \backslash \hat{F}$ and hence if $\hat{F}$ is an (M) peak face of X then F is a peak set of Ω.

THEOREM 3. *If F is a strong hull in Ω then Z is decomposable at the closed face* $\operatorname{conv}(\hat{F} \cup -i\hat{F})$.

PROOF. Let $G = N \cap Z$. We show first that there is an $s > 0$ such that for each neighborhood U of G in Z and each $\epsilon > 0$ there is an $h \in A(Z)$ satisfying

$$h \equiv 0 \quad \text{on } N, \quad \|h\| \leqq s, \quad |h(y) - 1| < \epsilon \quad \text{for all } y \in (\operatorname{ext} Z)^- \backslash U.$$

Let U be a compact neighborhood of G. Then U contains F and $-iF$. Let $V = \phi^{-1}(U \cap iU)$. Then V is a neighborhood of F in Ω. If $z \in \operatorname{ext} Z \backslash U$ either $z = \phi x$ or $z = -i\phi x$ with $x \in \Omega \backslash V$. Since F is a strong hull there is $f \in M$ satisfying

$$f \equiv 0 \quad \text{on } F, \quad |f(y) - 1| < \epsilon/2 \quad \text{on } \Omega \backslash V, \quad \|f\| \leqq r.$$

Hence by taking $\theta(f + if)$ we obtain a function

$$h(U, \epsilon) \in A(Z)$$

which is close to 1 at both ϕx and $-i\phi x$ for $x \in \Omega / V$. In fact

$$h \equiv 0 \quad \text{on } N, \qquad \|h\| \leqq 2r = s$$

and $|h(z) - 1| < \epsilon$ for all $z \in (\operatorname{ext} Z \backslash U)^-$ and hence for all $z \in (\operatorname{ext} Z)^- \backslash U$. (Since U is compact, $(\operatorname{ext} Z)^- \backslash U \subset (\operatorname{ext} Z \backslash U)^-$.) Thus $h(U, \epsilon)$ can be considered as a net in the ball of radius s of $A(Z)^{**}$. Let h_0 be a weak* limit point of this net. Then $h_0 \equiv 0$ on N.

Let $y \in Z$. By the Krein-Milman Theorem (see for example [5]) there is a probability measure μ_y on Z with $\operatorname{supp} \mu_y \subset (\operatorname{ext} Z)^-$ and

$$\int h d\mu_y = h(y) \quad \text{for all } h \in A(Z).$$

If $\mu_y(G) = 1$ then $y \in G$ and hence $h_0(y) = 0$. If $\mu_y(G) = 0$ we claim $h_0(y) = 1$. For given any $\delta > 0$ let U_0 be a neighborhood of G such that $\mu_y(U_0) < \delta/2(s+1)$ and let $\epsilon_0 = \delta/2$. Then if U is a neighborhood of G such that $U \subset U_0$ and $\epsilon < \epsilon_0$

$$\begin{aligned} |h(U, \epsilon)(y) - 1| &\leqq \int_U |h(U, \epsilon) - 1| \, d\mu_y + \int_{(\operatorname{ext} Z)^- \backslash U} |h(U, \epsilon) - 1| \, d\mu_y \\ &\leqq (\|h(U, \epsilon)\| + 1)\mu_y(U) + \epsilon < \delta. \end{aligned}$$

Finally suppose $0 < \mu_y(G) < 1$ and let

$$\nu = (1/\mu_y(G))\mu_y\,|_G.$$

Then ν is a probability measure on G and hence there is $x \in G$ such that $h(x) = \int h d\nu$ for all $h \in A(Z)$. Let

$$\eta = [1/(1 - \mu_y(G))](\mu_y - \mu_y\,|_G).$$

Then η is a probability measure on Z such that $\eta(G) = 0$. Hence $\eta = \eta_z$ for some $z \in Z$ such that $h_0(z) = 1$. Furthermore

$$\mu_y = \mu_y(G)\nu + (1 - \mu_y(G))\eta_z$$

and therefore $y = \mu_y(G)x + (1 - \mu_y(G))z$. Thus Z is decomposable at G under h_0.

Since in particular G is a face of Z,

$$G = \operatorname{conv}[(G \cap X) \cup (G \cap -iX)] = \operatorname{conv}(\hat{F} \cup -i\hat{F}).$$

COROLLARY 4. *If F is a strong hull in Ω then F is a peak set (within G_δ's) with respect to M.*

Suppose now M is a function *algebra*. Let $B = \phi^{-1}(\text{ext } X)$ be the Choquet boundary of Ω. Bishop's Theorem [**3**] that each point in B is (within G_δ's) a peak point follows from the fact that each point in B is a strong hull. This is essentially the $1/4 - 3/4$ Theorem [**3**] which in turn is a consequence of the following fact.

PROPOSITION 5. *Let F be a closed face of the compact convex set X which is a subset of the Hausdorff locally convex space E. If U is a neighborhood of F in X and ϵ, r are positive numbers there is an $f \in E^* + R$ such that*

(1) $f \geqq 0$ *on* X,

(2) $f|_F < \epsilon$,

(3) $f \geqq r$ *on* $X \setminus U$.

PROOF. It is well known that there is a compact convex set K in X such that $K \cap F = \emptyset$ and $X \setminus U \subset K$. Let

$$H = \operatorname{conv}[(X \times \{0\}) \cup (K \times \{r\})] \quad \text{in } E^* \times R.$$

Let G be a hyperplane in $E^* \times R$ separating H from $F \times \{\epsilon\}$. Then G is the graph of the desired function.

THEOREM 6. *If M is an algebra and $x \in B$ then for each neighborhood U of $x \in B$ and each $1 > \epsilon > 0$ there is an $h \in M$ such that*

$$\|h\| \leqq 1, \qquad h(x) > 1 - \epsilon, \qquad |h(y)| < \epsilon \quad \text{for all } y \in \Omega \setminus U.$$

PROOF. Let V be a neighborhood of $\phi(x)$ disjoint from $\phi(\Omega \setminus U)$. By

Proposition 5 there is $g \in M$ such that

$$\theta g \geqq 0 \quad \text{on } X, \qquad \theta g(x) < -\ln(1-\epsilon),$$
$$\theta g(y) \geqq -\ln \epsilon \quad \text{for all } y \in X \backslash V.$$

Let $h = ce^{-g}$, where $c = \exp(i \operatorname{Im} g(x))$.

COROLLARY 7. *If M is an algebra and $x \in B$ then $\{x\}$ is a strong hull and hence a peak point of M (within G_δ's).*

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UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024