DECOMPOSABLE COMPACT CONVEX SETS AND PEAK SETS FOR FUNCTION SPACES

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Abstract. Geometric conditions are known under which a closed face of a compact convex set is a peak set with respect to the space of continuous affine (real-valued) functions. The purpose of this note is to give an application of this "abstract-geometric" set-up to the problem of finding peak sets (or points) in a compact Hausdorff space with respect to a closed subspace of continuous complex-valued functions. In this fashion we obtain the strong hull criteria of Curtis and Figá-Talamanca and in particular the Bishop peak point theorem for function algebras.

Let $X$ be a compact convex subset of a Hausdorff locally convex space and let $A(X)$ denote the space of continuous real-valued affine functions on $X$. Then $A(X)$ is a Banach space under the supremum norm. We make the usual identification of $X$ with the set of positive normalized functionals in $A(X)^*$ with the weak* topology. If $P$ is a closed face of $X$ we say $X$ is decomposable at $P$ under $f$ if for any $G_s$ set $G$ in $X$ containing $F$ there is a nonnegative $h \in A(X)$ such that $f \in \{x \in X : h(x) = 0\}$.

We say $F$ is an $(A)$ peak face of $X$ (within $G_s$'s) if for any $G_s$ set $G$ in $X$ containing $F$ there is a nonnegative $h \in A(X)$ such that $F \subset \{x \in X : h(x) = 0\} \subset G$.

Our basic tool in what follows is the fact that if $X$ is decomposable at $F$ then $F$ is an $(A)$ peak face of $X$ (within $G_s$'s) [1].

Let $\Omega$ be a compact Hausdorff space and let $M$ be a uniformly closed separating subspace of continuous complex-valued functions (including the constants). Let $X$ be the state space of $M$, i.e., $X = \{x \in M^* : \|x\| = 1 = \langle 1, x \rangle\}$. Then $X$ is compact and convex in the weak* topology. If $\text{ext} X$ denotes the set of extreme points of $X$ then $\Omega$ is homeomorphic (under the evaluation map $\phi$) to $(\text{ext} X)^{-}$. If $x \in \Omega$ and $f \in M$ we write $f(x)$ for $\langle f, \phi x \rangle$. If $f \in M$ then $\text{Re}(f)$ is in $A(X)$. We deal with the real and imaginary parts of $f$ by considering...
in place of \( X \) the compact convex set
\[
Z \equiv \text{conv}(X \cup -iX).
\]

For each \( f \in M \) let \( \theta f \in A(Z) \) be defined by
\[
\theta f(z) = \text{Re}(f, z).
\]
Then \( \theta f(-iz) = \text{Re}(f, -iz) = \text{Im}(f, z) \).

**Proposition 1.** The map \( \theta : M \to A(Z) \) is an isomorphism (bounded, one-to-one and onto).

**Proof.** Let \( h \in M \) and choose \( x \in \text{ext } X \) such that \( |h(x)| = ||h|| = 1 \). Then either \( |\text{Re } h(x)| \geq \frac{1}{2} \) or \( |\text{Im } h(x)| \geq \frac{1}{2} \). Thus either \( |\theta h(x)| \geq \frac{1}{2} \) or \( |\theta h(-ix)| \geq \frac{1}{2} \). Hence
\[
||\theta h|| \leq ||h|| \leq 2||\theta h||.
\]
Since the range of \( \theta \) is always dense this shows \( \theta \) is onto.

We shall say the closed face \( F \) of \( X \) is an \((M)\) peak face (within \( G_s \)'s) if for each \( G \) set \( G \) in \( X \) containing \( F \) there is an \( f \in M \) such that \( ||f|| = 1 \) and
\[
F \subseteq \{ x \in X : f(x) = 1 \} \subseteq \{ x \in X : |f(x)| = 1 \} \subseteq G.
\]

**Theorem 2.** If \( F \) is a closed face of \( X \) such that \( Z \) is decomposable at \( \text{conv}(F \cup -iF) \) then \( F \) is an \((M)\) peak face of \( X \) (within \( G_s \)'s).

**Proof.** If \( G \) is a \( G_s \) of \( X \) containing \( F \) then by Proposition 1 there is an \( h \in M \) such that
\[
0 \leq \theta h \leq \sqrt{2}/2 \quad \text{on } Z, \quad \theta h = 0 \quad \text{on } \text{conv}(F \cup -iF),
\]
\[
0 < \theta h \quad \text{on } Z \backslash \text{conv}(G \cup -iG).
\]
Keeping in mind that \( \theta h(-iz) = \text{Im}(h, z) \) these properties say that
(1) \( h(X) \) is contained in the square inscribed in the first quadrant of the unit disk.
(2) \( h(F) = \{0\} \),
(3) if \( x \in X \backslash G \) then \( \Re(h, x), \Im(h, x) > 0 \). Thus \( 1 - e^{-\pi i/4}h \in M \) is easily seen to be a function of the required type.

Let \( F \) be a closed subset of \( \Omega \) and let \( F^\perp = \{ f \in M : f \equiv 0 \text{ on } F \} \). Let \( N \equiv F^{\perp \perp} \) be the polar of \( F^\perp \) in \( M^* \) and let \( F = N \cap X \).

Following Curtis and Figá-Talamanca [4] we say \( F \) is a **strong hull** if there is an \( r > 0 \) such that for any neighborhood \( V \) of \( F \) and any \( \epsilon > 0 \) there is an \( f \in M \) such that
\[
f \equiv 0 \quad \text{on } F, \quad |f(y) - 1| < \epsilon \quad \text{for } y \in \Omega \backslash V, \quad ||f|| \leq r.
\]
Let $F$ be a strong hull. Then for any $x \in \Omega \setminus F$ we have $\phi x \in X \setminus \hat{F}$ and hence if $\hat{F}$ is an $(M)$ peak face of $X$ then $F$ is a peak set of $\Omega$.

**Theorem 3.** *If $F$ is a strong hull in $\Omega$ then $Z$ is decomposable at the closed face $\text{conv}(\hat{F} \cup -i\hat{F})$.***

**Proof.** Let $G = N \cap Z$. We show first that there is an $s > 0$ such that for each neighborhood $U$ of $G$ in $Z$ and each $\epsilon > 0$ there is an $h \in A(Z)$ satisfying

$$h \equiv 0 \quad \text{on} \quad N, \quad \|h\| \leq s, \quad |h(y) - 1| < \epsilon \quad \text{for all} \quad y \in (\text{ext } Z) \setminus U.$$ 

Let $U$ be a compact neighborhood of $G$. Then $U$ contains $F$ and $-iF$. Let $V = \phi^{-1}(U \cap iU)$. Then $V$ is a neighborhood of $F$ in $\Omega$. If $z \in \text{ext } Z \setminus U$ either $z = \phi x$ or $z = -i\phi x$ with $x \in \Omega \setminus V$. Since $F$ is a strong hull there is $f \in M$ satisfying

$$f \equiv 0 \quad \text{on} \quad F, \quad |f(y) - 1| < \epsilon/2 \quad \text{on} \quad \Omega \setminus V, \quad \|f\| \leq r.$$ 

Hence by taking $\theta(f + if)$ we obtain a function

$$h(U, \epsilon) \in A(Z)$$ 

which is close to 1 at both $\phi x$ and $-i\phi x$ for $x \in \Omega / V$. In fact

$$h \equiv 0 \quad \text{on} \quad N, \quad \|h\| \leq 2r = s$$ 

and $|h(z) - 1| < \epsilon$ for all $z \in (\text{ext } Z \setminus U)^{-}$ and hence for all $z \in (\text{ext } Z) \setminus U$. (Since $U$ is compact, $(\text{ext } Z) \setminus U \subset (\text{ext } Z \setminus U)^{-}$.) Thus $h(U, \epsilon)$ can be considered as a net in the ball of radius $s$ of $A(Z)^{**}$. Let $h_{0}$ be a weak* limit point of this net. Then $h_{0} \equiv 0$ on $N$.

Let $y \in Z$. By the Krein-Milman Theorem (see for example [5]) there is a probability measure $\mu_{y}$ on $Z$ with $\text{supp } \mu_{y} \subset (\text{ext } Z)^{-}$ and

$$\int h d\mu_{y} = h(y) \quad \text{for all} \quad h \in A(Z).$$

If $\mu_{y}(G) = 1$ then $y \in G$ and hence $h_{0}(y) = 0$. If $\mu_{y}(G) = 0$ we claim $h_{0}(y) = 1$. For given any $\delta > 0$ let $U_{0}$ be a neighborhood of $G$ such that $\mu_{y}(U_{0}) < \delta / (s + 1)$ and let $\epsilon_{0} = \delta / 2$. Then if $U$ is a neighborhood of $G$ such that $U \subset U_{0}$ and $\epsilon < \epsilon_{0}$

$$|h(U, \epsilon)(y) - 1| \leq \int_{U} |h(U, \epsilon) - 1| d\mu_{y} + \int_{(\text{ext } Z) \setminus U} |h(U, \epsilon) - 1| d\mu_{y} \leq (\|h(U, \epsilon)\| + 1)\mu_{y}(U) + \epsilon < \delta.$$ 

Finally suppose $0 < \mu_{y}(G) < 1$ and let
\[
\nu = (1/\mu_y(G))\mu_y|_G.
\]

Then \(\nu\) is a probability measure on \(G\) and hence there is \(x \in G\) such that \(h(x) = \int h d\nu\) for all \(h \in A(Z)\). Let
\[
\eta = \left(1/(1 - \mu_y(G))\right)(\mu_y - \mu_y|_G).
\]

Then \(\eta\) is a probability measure on \(Z\) such that \(\eta(G) = 0\). Hence there is \(z \in Z\) such that \(h_0(z) = 1\). Furthermore
\[
\mu_y = \mu_y(G)x + (1 - \mu_y(G))\eta_z
\]
and therefore \(\eta = \mu_y(G)x + (1 - \mu_y(G))z\). Thus \(Z\) is decomposable at \(G\) under \(h_0\).

Since in particular \(G\) is a face of \(Z\),
\[
G = \text{conv}\{(G \cap X) \cup (G \cap -iX)\} = \text{conv}(\tilde{F} \cup i\tilde{F}).
\]

**Corollary 4.** If \(F\) is a strong hull in \(\Omega\) then \(F\) is a peak set (within \(G_i\)'s) with respect to \(M\).

Suppose now \(M\) is a function algebra. Let \(B = \phi^{-1}(\text{ext } X)\) be the Choquet boundary of \(\Omega\). Bishop's Theorem [3] that each point in \(B\) is (within \(G_i\)'s) a peak point follows from the fact that each point in \(B\) is a strong hull. This is essentially the 1/4 - 3/4 Theorem [3] which in turn is a consequence of the following fact.

**Proposition 5.** Let \(F\) be a closed face of the compact convex set \(X\) which is a subset of the Hausdorff locally convex space \(E\). If \(U\) is a neighborhood of \(F\) in \(X\) and \(\epsilon, r\) are positive numbers there is an \(f \in E^* + R\) such that

1. \(f \geq 0\) on \(X\),
2. \(f|_F < \epsilon\),
3. \(f \geq r\) on \(X \setminus U\).

**Proof.** It is well known that there is a compact convex set \(K\) in \(X\) such that \(K \cap F = \emptyset\) and \(X \setminus U \subset K\). Let
\[
H = \text{conv}\{(X \times \{0\}) \cup (K \times \{r\})\} \text{ in } E^* \times R.
\]
Let \(G\) be a hyperplane in \(E^* \times R\) separating \(H\) from \(F \times \{\epsilon\}\). Then \(G\) is the graph of the desired function.

**Theorem 6.** If \(M\) is an algebra and \(x \in B\) then for each neighborhood \(U\) of \(x \in B\) and each \(1 > \epsilon > 0\) there is an \(h \in M\) such that
\[
\|h\| \leq 1, \quad h(x) > 1 - \epsilon, \quad |h(y)| < \epsilon \quad \text{for all } y \in \Omega \setminus U.
\]

**Proof.** Let \(V\) be a neighborhood of \(\phi(x)\) disjoint from \(\phi(\Omega \setminus U)\). By
Proposition 5 there is $g \in M$ such that

$$\theta g \geq 0 \quad \text{on } X, \quad \theta g(x) < -\ln(1 - \epsilon),$$

$$\theta g(y) \geq -\ln \epsilon \quad \text{for all } y \in X \setminus V.$$

Let $h = ce^{-\omega}$, where $c = \exp(i \text{Im } g(x))$.

**Corollary 7.** If $M$ is an algebra and $x \in B$ then $\{x\}$ is a strong hull and hence a peak point of $M$ (within $G_i$'s).

**References**


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