NOTE ON THE SHAPIRO POLYNOMIALS

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1. **Introduction.** The polynomials $P_n(x)$ and $Q_n(x)$, which we are concerned with here, were introduced in 1951 by H. S. Shapiro [5, p. 39] in his study of the magnitude of certain trigonometric sums. They are defined recursively by the formulas

$$(1)\quad P_{n+1}(x) = P_n(x) + x^{2^n}Q_n(x), \qquad Q_{n+1}(x) = P_n(x) - x^{2^n}Q_n(x),$$

where $n \geqq 0$ and $P_0(x) = Q_0(x) = 1$. (See [4] also. Note in this reference that $P_0(x) = Q_0(x) = x$.)

These polynomials have been used by Kahane and Salem in their book [1] to prove several theorems about trigonometric series. Rider [2] used a generalization of these polynomials to complete the solution of a problem partially solved in [4]. In a more recent paper Rider [3] employed the polynomials to exhibit certain subalgebras of the group algebra of the unit circle. In particular, in this paper Rider obtained a special case of Theorem 4 below.

The first few polynomials are

$$P_1(x) = 1 + x, \qquad P_2(x) = 1 + x + x^2 - x^3,$$
$$P_3(x) = 1 + x + x^2 - x^3 + x^4 + x^5 - x^6 + x^7,$$
$$Q_1(x) = 1 - x, \qquad Q_2(x) = 1 + x - x^2 + x^3,$$
$$Q_3(x) = 1 + x + x^2 - x^3 - x^4 - x^5 + x^6 - x^7.$$

It is clear from this definition that $\deg P_n = \deg Q_n = 2^n - 1$.

In this note we will derive a relation between $P_n(x)$ and $Q_n(x)$ and use it to show that these polynomials have equal discriminants. We will also find a formula for the resultant of the two polynomials, and develop an explicit formula for their coefficients. The latter will then be used to compute the value of $P_n(x)$ at $x = \pm 1$, $\pm i$, and certain other points on the unit circle.

2. We begin by deriving the relation that exists between $P_n(x)$ and $Q_n(x)$.

THEOREM 1. $Q_n(x) = (-1)^n x^{2^n - 1} P_n(-1/x)$, $n \geqq 0$.

PROOF. By induction. The theorem holds for $n = 0, 1$. Assume the relation for n, $n \geqq 1$. Then

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$$\begin{aligned}(-1)^{n+1}x^{2^{n+1}-1}P_{n+1}(-1/x)\\ &= (-1)^{n+1}x^{2^{n+1}-1}[P_n(-1/x) + (-1/x)^{2^n}Q_n(-1/x)]\\ &= (-1)^{n+1}x^{2^{n+1}-1}[(-1)^nQ_n(x)/x^{2^n-1} + x^{-2^n}(-1)^n(-1/x)^{2^n-1}P_n(x)]\\ &= P_n(x) - x^{2^n}Q_n(x) = Q_{n+1}(x). \blacksquare\end{aligned}$$

The following properties of the discriminant D of a polynomial will be of use in establishing the corollary below. Let $c\neq 0$ be a constant and $f(x)$ by a polynomial of degree n. Then

(i) $D(f(cx))=c^{n(n-1)}D(f(x))$.

(ii) $D(cf(x))=c^{2n-2}D(f(x))$.

(iii) $D(x^nf(1/x))=D(f(x))$.

COROLLARY. $D(P_n(x))=D(Q_n(x))$, $n\geqq 0$.

PROOF.

$$\begin{aligned}D(Q_n(x)) &= D((-1)^nx^{2^n-1}P_n(-1/x))\\ &= D((-1)^{n+1}x^{2^n-1}P_n(1/x)),\end{aligned}$$

using (i) with $c=-1$. The corollary then follows from (ii) and (iii). $\blacksquare$ The first few completely factored values of $D(P_n(x))$ are listed in the table below

n	$D(P_n(x))$
1	1
2	$-2^2\cdot 11$
3	$2^{10}\cdot 5^2\cdot 193$
4	$2^{34}\cdot 32834009652827$

We next recall several properties of the resultant R of two polynomials f and g of degree n and m respectively.

(i) $R(f, cg)=c^nR(f, g)$, c a constant.

(ii) $R(f, g)=a^dR(f, g+\lambda f)$, where a is the leading coefficient of f, λ is an arbitrary polynomial, and $d=\deg g-\deg (g+\lambda f)$.

(iii) $R(f, g)=(-1)^{mn}R(g, f)$.

(iv) $R(f, gh)=R(f, g)R(f, h)$.

THEOREM 2. $R(P_n(x), Q_n(x))=(-1)^{n-1}2^{2^{n+1}-n-2}$, $n\geqq 1$.

PROOF. For $n=1$ we have $R(P_1, Q_1)=2$. Suppose $n>1$. Then

$$\begin{aligned} R(P_n, Q_n) &= R(P_{n-1} + x^{2^{n-1}}Q_{n-1}, P_{n-1} - x^{2^{n-1}}Q_{n-1}) \\ &= R(P_{n-1} + x^{2^{n-1}}Q_{n-1}, 2P_{n-1}) \\ &= -2^{2^n-1}R(P_{n-1}, P_{n-1} + x^{2^{n-1}}Q_{n-1}) \\ &= -2^{2^n-1}R(P_{n-1}, x^{2^{n-1}}Q_{n-1}) \\ &= -2^{2^n-1}R(P_{n-1}, x^{2^{n-1}})R(P_{n-1}, Q_{n-1}). \end{aligned}$$

But $R(P_{n-1}, x^{2^{n-1}})=1$. Hence $R(P_n, Q_n) = -2^{2^n-1}R(P_{n-1}, Q_{n-1})$. From this reduction step, used repeatedly, we obtain the evaluation $R(P_n, Q_n) = \{\prod_{s=2}^n(-2^{2^s-1})\}R(P_1, Q_1) = (-1)^{n-1}2^{2^{n+1}-n-2}$. ■

The next theorem permits the generation of $P_n(x)$ and $Q_n(x)$ without combining the two types of polynomials.

THEOREM 3.

$$\begin{aligned} P_{n+1}(x) &= P_n(x^2) + xP_n(-x^2), \qquad n \geqq 0. \\ Q_{n+1}(x) &= Q_n(x^2) + xQ_n(-x^2), \qquad n \geqq 1. \end{aligned}$$

PROOF. By induction. The formulas are true for $n=0$, 1. Assume *both* formulas hold for n, $n \geqq 1$. Then

$$\begin{aligned} P_{n+1}(x) &= P_n(x) + x^{2^n}Q_n(x) \\ &= [P_{n-1}(x^2) + xP_{n-1}(-x^2)] + x^{2^n}[Q_{n-1}(x^2) + xQ_{n-1}(-x^2)] \\ &= [P_{n-1}(x^2) + x^{2^n}Q_{n-1}(x^2)] + x[P_{n-1}(-x^2) + x^{2^n}Q_{n-1}(-x^2)]. \end{aligned}$$

Hence,

$$P_{n+1}(x) = P_n(x^2) + xP_n(-x^2). \tag{2}$$

The formula for $Q_{n+1}(x)$ is established in a similar manner. ■

3. We now turn to an investigation of the coefficients of $P_n(x)$. (The corresponding results can be obtained for $Q_n(x)$ through the use of Theorem 1.)

It is clear from (1) that $P_n(x)$ has coefficients ± 1, without gaps, and that the first 2^n coefficients of $P_{n+1}(x)$ are identical with those of $P_n(x)$. It follows then that these coefficients do not depend on n, so we can write $P_n(x) = \sum_{r=0}^{2^n-1} a(r)x^r$, $n \geqq 0$. (We may, of course, also consider $P_n(x)$ as the first 2^n terms of the infinite series $P_\infty(x) = \sum_{r=0}^{\infty} a(r)x^r$.)

We will now derive an explicit formula for $a(r)$.

THEOREM 4. *If we write* $r = r_0 + r_1 \cdot 2 + r_2 \cdot 2^2 + \cdots + r_k \cdot 2^k$, $k \geqq 0$, $r_i = 0$ *or* 1, *then*

$$a(r) = (-1)^{r_0r_1+r_1r_2+\cdots+r_{k-1}r_k}. \tag{3}$$

PROOF. We observe in (2) that the even and odd degree terms on the right side are separated, which allows us to equate coefficients, obtaining the relations $a(2r)=a(r)$ and $a(2r+1)=(-1)^r a(r)$. If we write $a(r)=(-1)^{e(r)}$, then

$$e(2r) \equiv e(r) \quad \text{and} \quad e(2r+1) \equiv r + e(r) \pmod 2. \tag{4}$$

Proceeding by induction on k, we verify for $k=0$ that $1=a(r_0)=(-1)^0$, where $r_0=0$ or 1. Assume next that $e(r)=r_1r_2+r_2r_3+\cdots+r_{k-1}r_k$ for any $r=r_1+r_2\cdot 2+\cdots+r_k\cdot 2^{k-1}$ of k digits. Consider the number $2r+r_0$, where $r_0=0$ or 1. Then using (4) $e(2r+r_0)\equiv r_0r+e(r)\equiv r_0r_1+e(r)=r_0r_1+r_1r_2+\cdots+r_{k-1}r_k \pmod 2$. $\blacksquare$ (Note the particular case $a(2^t)=1$.)

4. We next consider the problem of evaluating $P_n(x)$ at certain points on the unit circle. We begin with

THEOREM 5.

$$P_{2n}(1) = 2^n, \qquad P_{2n+1}(1) = 2^{n+1}, \qquad n \geqq 0.$$
$$P_{2n}(-1) = 2^n, \qquad P_{2n+1}(-1) = 0, \qquad n \geqq 0.$$

PROOF. Let $\theta(n)$ be the number of $a(r)$ in $P_n(x)$ that are positive. In particular, let $\theta_0(n)$ be the number of $a(2r)$ and $\theta_1(n)$ be the number of $a(2r+1)$ in $P_n(x)$ that are positive. Then certainly

$$\theta(n) = \theta_0(n) + \theta_1(n). \tag{5}$$

Since the first term on the right side of (2) contains all the terms of even degree, we have

$$\theta_0(n+1) = \theta(n), \tag{6}$$

and hence by (5)

$$\theta_0(n+1) = \theta_0(n) + \theta_1(n). \tag{7}$$

Also, since the second term on the right side of (2) can be written as $\sum_{r=0}^{2^n-1}(-1)^r a(r)x^{2r+1}$, we find that

$$\theta_1(n+1) = \theta_0(n) + [2^{n-1} - \theta_1(n)].$$

Adding this equation to (7), and using (5), we obtain

$$\theta(n+1) = \theta_0(n+1) + \theta_1(n+1) = 2\theta_0(n) + 2^{n-1}.$$

Finally, from (6) we derive the recursion relation

$$\theta(n+1) = 2\theta(n-1) + 2^{n-1}.$$

With the initial conditions $\theta(0)=1$, and $\theta(1)=2$, the solution is readily found to be

$$\text{(8)} \qquad \theta(2n) = 2^{2n-1} + 2^{n-1}, \qquad \theta(2n+1) = 2^{2n} + 2^n, \quad n \geqq 0.$$

From the equation $P_n(1) = \theta(n) - [2^n - \theta(n)] = 2\theta(n) - 2^n$, we conclude that $P_{2n}(1) = 2^n$ and $P_{2n+1}(1) = 2^{n+1}$. If we now set $x=1$ in (2), we have $P_n(-1) = P_{n+1}(1) - P_n(1)$, whence $P_{2n}(-1) = 2^n$ and $P_{2n+1}(-1) = 0$. ∎

With a knowledge of $P_n(\pm 1)$, we are in a position to find the values at $x = e^{\pi i/2^t}$. For example, setting $x=i$ in (2), we obtain $P_{n+1}(i) = P_n(-1) + iP_n(1)$, whence $P_{2n}(i) = i \cdot 2^n$ and $P_{2n+1}(i) = (1+i)2^n$. The values at $x = -i$ are found by conjugating.

REMARK. It can readily be shown by repeated use of (2) that the series $P_\infty(x)$ diverges at the dense set of points $\exp(2\pi r i/2^s)$ on the unit circle.

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