SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

THE TOEPLITZ-HAUSDORFF THEOREM
FOR LINEAR OPERATORS

KARL GUSTAFSON

Theorem (Toeplitz-Hausdorff). The numerical range \( W(A) = \{(Ax, x):\|x\| = 1, x \in D(A)\} \) of an arbitrary (perhaps unbounded and not densely defined) linear operator \( A \) in a (pre)-Hilbert space (real or complex) is convex.

Proof. Since \( W(\mu A + \gamma) = \mu W(A) + \gamma \), for scalars \( \mu, \gamma \), it suffices to consider the situation \((Ax_1, x_1) = 0, (Ax_2, x_2) = 1, \|x_i\| = 1, x_i \in D(A), i = 1, 2\). Let \( x = \alpha x_1 + \beta x_2, \alpha \) and \( \beta \) real, and require

\[
(1) \quad \|x\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta \text{ Re}(x_1, x_2) = 1,
\]

and desire (for each \( 0 < \lambda < 1 \))

\[
(2) \quad (Ax, x) = \beta^2 + \alpha \beta \{ (Ax_1, x_2) + (Ax_2, x_1) \} = \lambda.
\]

Let \( B = (Ax_1, x_2) + (Ax_2, x_1) \); if \( B \) is real, then the system (1), (2) describes an ellipse (intercepts \( \pm 1, \pm 1 \)) and a hyperbola (intercepts \( \pm \lambda^{1/2} \)) and clearly possesses (four, since \( |\text{Re}(x_1, x_2)| < 1 \) by Schwarz’s Inequality) solutions. But \( B \) can always be guaranteed real by using an appropriate (scalar multiple of) \( x_1 \); i.e., explicitly, use \( x'_1 = \mu x_1 \), where \( \mu = a + ib \) satisfies \( (1') \|\mu\|^2 = a^2 + b^2 = 1 \) and \( (2') \text{ Im } B(x'_1) = a \text{ Im } B(x_1) + b \text{ Re } \{(Ax_1, x_2) - (Ax_2, x_1)\} = 0 \), a system clearly possessing (two) solutions.

Remarks. For background and (some) previous proofs, see [1]–[6]; our approach (used in a less simple way in [7], from which the demonstration above evolved) of continuity arguments utilizing conic sections is also present in [6] and is certainly not new as a general technique.

References


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University of Colorado, Boulder, Colorado 80302