THE SPECTRUM OF A SELFADJOINT COMPRESSION OF A SELFADJOINT OPERATOR

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ABSTRACT. The relation between the spectrum of a selfadjoint operator and the spectrum of its compression is investigated. In particular, we show that the compression of a selfadjoint operator is essentially selfadjoint if and only if the spectrum of the closure of the compression is contained in the closed convex hull of the spectrum of the operator. Relations between two conceptions of compressions or projections of operators are also considered.

Let A be an operator on a Hilbert space H. $AP\sigma(A)$ will denote the approximate point spectrum of A, $\mathrm{co}\sigma(A)$ the closed convex hull of the spectrum of A, $W(A)$ the numerical range of A, $\overline{W}(A)$ the closed numerical range of A, and $\overline{A}$ the closure of A if A is preclosed. If S is a closed subspace of H and P is the orthogonal projection onto S, then the compression of A to S (defined by Halmos [2]), written $C(A)$ or $C_P(A)$, is defined by $C(A)x=PAx$ for x in the domain of $C(A)$, $D(C(A))$, which is $D(A)\cap S$. If $S\subseteq D(A)$ or $S^\perp\subseteq D(A)$, and $A=A^*$, then $C(A)=C(A)^*$ (see Stenger [7]). Recently a number of articles have appeared ([7], [3], [1], and [6]) rediscovering a result of Schechter [4] and [6] (that if A is closed and densely defined and if B is closed, densely defined with closed range of finite codimension then $(AB)^*=B^*A^*$) which provides an easy proof of the result that if S is of finite codimension and $A=A^*$ then $C(A)=C(A)^*$ (see Gustafson [1]).

Sometimes the operator PAP, as well as $C(A)$, is called the projection of A (as in [7], [1], and [5]). $C(A)$ is the restriction of PAP to a completely reducing subspace (S with projection P completely reduces an operator A if $PA\subseteq AP$) and thus if PAP is selfadjoint, then $C(A)$ is self adjoint. The converse also holds, as is seen in a following proposition. If S is only reducing, i.e. $S\cap D(A)$ and $S^\perp \cap D(A)$ are invariant under A, then the compression of A to S need not be selfadjoint, or even maximally symmetric, as is shown by the example where $H=L^2[0, 1]$, $A=id/dx$ with $D(A)=\{u\in H: u$ is absolutely continuous, $u'\in H$, and $u(0)=u(1)\}$. Let $S=\{u\in H: u=0$

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a.e. on $[0, \frac{1}{3}]$ and on $[\frac{2}{3}, 1]\}$. Then $C(A)$ is symmetric with deficiency indices (1, 1).

LEMMA. *If A is densely defined, P is the orthogonal projection onto S, and $PA \subseteq AP$, then $C(A^*) = C(A)^*$. If, in addition, A is normal then $C(A)$ is normal.*

PROOF. Let $x \in D(C(A)) = D(A) \cap S$, $y \in D(C(A^*)) = D(A^*) \cap S$. Then $(C(A)x, y) = (Ax, y) = (x, A^*y) = (x, C(A^*)y)$. Thus $C(A^*) \subseteq C(A)^*$. Let $y \in D(C(A)^*)$, $x \in D(A)$. Then $x = Px + (I-P)x$, so $(Ax, y) = (APx, y) + (A(I-P)x, y) = (APx, y) = (C(A)Px, y) = (x, C(A)^*y)$. Thus $y \in D(A^*) \cap S$ and $C(A^*) = C(A)^*$.

Assume A is normal. $C(A)$ is closed since S is invariant and using $C(A)^* = C(A^*)$ we have $C(A)C(A)^* = C(A)^*C(A)$ and thus $C(A)$ is normal. $\square$

PROPOSITION. *$C(A)$ is normal (or selfadjoint) if and only if PAP is normal (or selfadjoint).*

PROOF. If PAP is normal, then S is a completely reducing subspace, $C(PAP)$ is normal, and $C(PAP)^* = C((PAP)^*)$. Since $D(C(PAP)) = S \cap D(PAP) = S \cap D(A) = D(C(A))$ we have $C(A) = C(PAP)$ is normal.

If $C(A)$ is normal, PAP is densely defined and closed. Define an operator B on H by $Bx = C(A)^*Px$ for all x such that Px is in $D(A)$. Thus $D(B) = D(PAP)$. Let $y \in D(B)$ and $x \in D(PAP)$. Then $(PAPx, y) = (C(A)Px, y) = (Px, C(A)^*Py) = (x, C(A)^*Py) = (x, By)$. Thus $B \subseteq (PAP)^*$. Let $z \in D((PAP)^*)$ and $x \in D(C(A))$. Then $(C(A)x, Pz) = (PAPx, Pz) = (PAPx, z) = (x, (PAP)^*z)$. So $Pz \in D(C(A)^*) = D(C(A)) = S \cap D(A)$. Thus $B = (PAP)^*$. $D((PAP)(PAP)^*) = D((PAP)^*(PAP))$ and for $x \in D((PAP)(PAP)^*)$, $(PAP)(PAP)^*x = PAP\, Bx = C(A)C(A)^*Px = C(A)^*C(A)Px = (PAP)^*(PAP)x$. Thus PAP is normal.

If $C(A)$ or PAP is selfadjoint then PAP or $C(A)$ is normal and symmetric, and thus selfadjoint. $\square$

PROPOSITION. *If $A = A^*$, then $\sigma(\overline{C}(A)) \subseteq \mathrm{co}\ \sigma(A)$ if and only if $\overline{C}(A) = \overline{C}(A)^*$.*

PROOF. $C(A)$ is symmetric, so it is preclosed. Assume $\sigma(\overline{C}(A)) \subseteq \mathrm{co}\ \sigma(A) \subseteq \boldsymbol{R}$. $\overline{C}(A)$ is densely defined, since if $(x, y) = 0$ for all $x \in D(\overline{C}(A))$ and as there is an element $z \in D(\overline{C}(A))$ such that $(\overline{C}(A) + iI)z = y$, then $0 = (x, (\overline{C}(A) + iI)z) = ((C(A) - iI)x, z)$ for all $x \in D(C(A))$. Then $i \in \sigma(\overline{C}(A))$, contradiction. Thus the closed, densely defined symmetric operator $\overline{C}(A)$ has deficiency indices (0, 0), and is selfadjoint.

If $\overline{C}(A)=\overline{C}(A)^*$, then $\sigma(\overline{C}(A))=AP\sigma(\overline{C}(A))\subseteq\overline{W}(\overline{C}(A))\subseteq\overline{W}(C(A))\subseteq\overline{W}(A)=\text{co }\sigma(A)$. $\square$

The proposition almost holds for PAP:

PROPOSITION. *If $A=A^*$, then $(PAP)^-$ is selfadjoint if and only if $\sigma((PAP)^-)\subseteq\text{co }\sigma(A)\cup\{0\}$.*

PROOF. If $(PAP)^-$ is selfadjoint, then $\overline{C}(A)=C(A)^{**}=C(PAP)^{**}$ $C((PAP)^{**})=C((PAP)^-)$ is selfadjoint and $\sigma(C((PAP)^-))=\sigma(\overline{C}(A))\subseteq\text{co }\sigma(A)$. Thus $\sigma((PAP)^-)=\sigma(C_P((PAP)^-))\cup\sigma(C_{I-P}((PAP)^-))\subseteq\text{co }\sigma(A)\cup\{0\}$.

If $\sigma((PAP)^-)\subseteq\text{co }\sigma(A)\cup\{0\}$, then $(PAP)^-$ is a closed, densely defined symmetric operator with real spectrum, and thus is selfadjoint. $\square$

An example distinguishing the two cases is $H=\mathbf{C}^2$, $A=I$, $S=\{(x,\ y):x=y\}$. Then $\sigma(A)=\{1\}$, $C(A)=I$, $P(x,\ y)=((x+y)/2,\ (x+y)/2)$, and $PAP(1,\ -1)=0$, so $0\in\sigma(PAP)$.

In the case of $C(A)$, the inclusion may be proper, in fact we may have $\sigma(\overline{C}(A))\cap\sigma(A)=\varnothing$ as is shown by the example with

$$H=\mathbf{C}^2,\qquad A=\begin{vmatrix}1 & 0\\ 0 & -1\end{vmatrix},$$

and $S=\{(x,\ y):x=y\}$. Then $\sigma(A)=\{-1,\ 1\}$, $C(A)=0$, and $\sigma(C(A))=\{0\}$.

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