ASYMPTOTICS IN RANDOM (0, 1)-MATRICES

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ABSTRACT. Let $M^n(i)$ be the class of $n \times n$ (0, 1)-matrices with i ones. We wish to find the first and second moments of Perm B, the permanent of the matrix B, as B ranges over the class $M^n(i)$. We succeed for $i > n^{3/2+\epsilon}$ in finding an asymptotic estimate of these quantities. It turns out that the square of the first moment is asymptotic to the second moment, so we may conclude that almost all matrices in $M^n(i)$ have asymptotically the same permanent. It is suggested that the technique employed will also enable us to evaluate asymptotically the number of hamiltonian circuits in a random graph with i links on n vertices.

The field of number theory abounds in "probabilistic" results. Thus, the prime number theorem may be viewed in this way: for large n, the probability that the integer n is prime is $n/\log n$. Or, again, a "strong probabilistic" result: for large n, almost all integers n have about $\log \log n$ divisors. We value results such as these because it seems unlikely that an "exact" solution in "closed form" is attainable: e.g. that we could compute the function $\pi(n)$ in a bounded number of elementary operations k, for any integer n.

Combinatorial functions exist whose complexity makes them a suitable subject for "probabilistic" theorems. It is conjectured that the permanent of an $n \times n$ matrix A,

$$\text{Perm }(A) = \sum_{\pi \in S_n} \left(\prod_{i=1}^{n} A(i, \pi(i)) \right)$$

requires a number N of operations for its computation which is exponential in n. Compare the fact that there exists an algorithm requiring n^3 operations to compute $\det(A)$.

Some probabilistic results concerning graphs and (0, 1)-matrices have been proven by P. Erdös and A. Rényi. An example: almost all graphs with $cn \log n$ edges on n vertices are connected. It is a conjecture that almost all $n \times n$ (0, 1)-matrices with at least $n^{1+\epsilon}$ ones, $\epsilon > 0$, have a nonzero permanent.

In this paper, working in a different range, where the number of ones is a nontrivial proportion of the n^2 spaces of a matrix, a strong probabilistic result is obtained. The "average" permanent is derived

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and it is shown that almost all matrices have permanents asymptotic to this average. We begin with some definitions.

Let $M^n(i)$ be the class of $n\times n$ (0, 1)-matrices with i ones. We wish to find asymptotic estimates of the quantities

$$\operatorname*{Average}_{B\in M^n(i)} (\text{Perm } B) \qquad \text{and} \qquad \operatorname*{Average}_{B\in M^n(i)} (\text{Perm } B)^2.$$

Choose $B\in M^n(i)$ at random; also choose an element π from the permutation group on n elements.

DEFINITION. Let x_π^B be a random variable dependent on B and π, given by

$$\begin{aligned} x_\pi^B &= 1 \qquad \text{iff } B(j,k) = 1 \quad \text{when } \pi(j) = k, \quad \text{for all } j = 1(1)n, \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

i.e., x_π^B is 1 iff the permutation π gives a contribution of 1 to Perm (B). Now define

$$X^B = \text{Perm } B = \sum_{\pi\in S_n} x_\pi^B. \tag{1.1}$$

Obviously X^B is a random variable dependent only on B which takes on the value Perm B.

Given π, we may calculate $P(x_\pi^B=1)$ by enumerating the number of matrices B for which $x_\pi^B=1$ and dividing by the total number of matrices $B\in M^n(i)$.

A matrix B such that $x_\pi^B=1$ has 1's in all positions (j, k) where $\pi(j)=k$. The remaining 1's may be placed anywhere and this may be done in $\binom{n^2-n}{i-n}$ ways.

The total number of matrices $B\in M^n(i)$, $\#M^n(i)$, is given by $\binom{n^2}{i}$. Therefore, for any $\pi\in S_n$,

$$P(x_\pi^B=1)=\binom{n^2-n}{i-n}\bigg/\binom{n^2}{i}=\prod_{j=0}^{n-1}(i-j)\bigg/\prod_{j=0}^{n-1}(n^2-j). \tag{1.2}$$

LEMMA 1. *Asymptotically in i, with $m\leq i$, $m=o(i)$,*

$$\prod_{j=0}^{m-1}(i-j) = i^m \exp\left(\frac{-m(m-1)}{2i}\right)\left(1+O\left(\frac{m^3}{i^2}\right)\right). \tag{1.3}$$

PROOF. We have

$$\log\left(\prod_{j=0}^{m-1}(i-j)\right) = \sum_{j=0}^{m-1}\log(i-j) = m\log i + \sum_{j=0}^{m-1}\log(1-j/i). \tag{1.4}$$

We may expand $\log(1-j/i)$, in a Taylor series with remainder, as

$$\log(1-j/i) = -j/i - R_2(j/i),$$

where $R_2(j/i)$ is strictly positive and may be bounded by

$$R_2(j/i) \leqq (j/i)^2/2(1-j/i)^2.$$

Note that

$$\max_{0\leqq j\leqq m-1} \left| R_2(j/i) \right| \leqq (m/i)^2/2(1-m/i)^2.$$

Therefore the total absolute error incurred in the sum $\sum_{j=0}^{m-1}\log(n-j/i)$ by substituting $-j/i$ for $\log(1-j/i)$ is bounded by

$$\frac{m^3}{i^2}\left(\frac{1}{2(1-m/j)^2}\right) = O\left(\frac{m^3}{i^2}\right)$$

and is strictly negative. Equation (1.4) gives

$$\log\left(\prod_{j=0}^{m-1}(i-j)\right) = m\log i - \sum_{j=0}^{m-1} j/i - O\left(\frac{m^3}{i^2}\right). \tag{1.5}$$

Since $\sum_{j=0}^{m-1} j/i = m(m-1)/2i$, exponentiating equation (1.5) proves the lemma.

THEOREM I. *Asymptotically in n, with $i > n^{3/2+\epsilon}$,*

$$\operatorname*{Average}_{B\in M^n(i)} \text{Perm } B = n!(i/n^2)^n \exp(-\tfrac{1}{2}(n^2/i-1))(1+o(n^{-2\epsilon})).$$

PROOF. We evaluate $P(x_\pi^B = 1)$ as given in equation (1.2). Applying Lemma 1.1, we see

$$\prod_{j=0}^{n-1}(i-j) = i^n \exp(-n^2/2i)(1+o(n^{-2\epsilon}))$$

and

$$\prod_{j=0}^{n-1}(n^2-j) = n^{2n}e^{-1/2}(1+o(n^{-1})).$$

Therefore

$$P(x_\pi^B = 1) = (i/n^2)^n \exp(-\tfrac{1}{2}(n^2/i-1))(1+o(n^{-2\epsilon})).$$

By equation (1.1), since $P(x_\pi^B = 1) = E(x_\pi^B)$, we have

$$\underset{B\in M^n(i)}{\text{Average}}\ \text{Perm}\, B = E(X^B) = \sum_{\pi\in S_n} E(x_\pi^B)$$
$$= n!\left(\frac{i}{n^2}\right)^n \exp(-\tfrac{1}{2}(n^2/i - 1))(1 + o(n^{-2\epsilon}))$$

which completes the proof.

We now turn to the evaluation of the second moment. The squared permanent of a randomly chosen matrix $B\in M^n(i)$ is given by:

$$\text{(1.6)} \qquad (\text{Perm}\, B)^2 = \sum_{\pi\in S_n} x_\pi^B \sum_{\sigma\in S_n} x_\sigma^B = \sum_{\sigma\in S_n}\sum_{\pi\in S_n} x_\pi^B \cdot x_\sigma^B.$$

Consider the ordered pairs $(\pi, \sigma)\in S_n\times S_n$. We may partition $S_n\times S_n$ as follows:

$$\text{(1.7)} \qquad S_n \times S_n = B_0 + B_1 + B_2 + \cdots + B_n,$$

where $(\pi, \sigma)\in B_k$ iff $\pi(j)=\sigma(j)$ for exactly k integers j in $\{1, 2, \cdots, n\}$.

We may estimate the number of elements (π, σ) in block B_k. The permutation π may be chosen in any of $n!$ ways. Now for each fixed π, the expanded problème des récontres gives us an exact expression for the number of permutations σ, $\#\sigma$, such that $(\pi, \sigma)\in B_k$.

$$\#\sigma = \frac{1}{k!}\, n!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-k}}{(n-k)!}\right).$$

We conclude

$$\text{(1.8)} \quad \#B_k = \frac{1}{k!}\,(n!)^2\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-k}}{(n-k)!}\right).$$

Therefore, asymptotically in $n-k$, (1.7) gives

$$\text{(1.9)} \qquad \#B_k = \frac{1}{k!}\,(n!)^2\, e^{-1}\left(1 + O\left(\frac{1}{(n-k+1)!}\right)\right).$$

Note especially that for all k, $\#B_k \leqq (1/k!)(n!)^2$.

By the same method used to calculate $P(x_\pi^B=1)$, we find, for $(\pi, \sigma)\in B_k$,

$$\text{(1.10)} \qquad P(x_\pi^B\cdot x_\sigma^B = 1) = \prod_{j=0}^{2n-k-1}(i-j) \Big/ \prod_{j=0}^{2n-k-1}(n^2-j).$$

THEOREM II. *Let $i>n^{3/2+\epsilon}$. Then asymptotically with n,*

$$\operatorname*{Average}_{B\in M^n(i)} (\text{Perm } B)^2$$
$$= (n!)^2(i/n^2)^{2n}\exp(-(n^2/i-1))(1+O(n^{-1/4-\epsilon})+O(n^{-2\epsilon})).$$

PROOF. By (1.6) and (1.7), we have the random variable $(\text{Perm } B)^2$, where B is chosen at random from $M^n(i)$, given by

$$(\text{Perm } B)^2 = \sum_{k=0}^{n}\left(\sum_{(\pi,\sigma)\in B_k} x_\pi^B\cdot x_\sigma^B\right).$$

Since $P(x_\pi^B\cdot x_\sigma^B=1)=E(x_\pi^B x_\sigma^B)$, and since this quantity is independent of the exact $(\pi,\sigma)\in B_k$, we have

$$\operatorname*{Average}_{B\in M^n(i)} (\text{Perm } B)^2 = E((\text{Perm } B)^2)$$
$$(1.11)\qquad = \sum_{k=0}^{n}(\#B_k)\cdot P(x_\pi^B\cdot x_\sigma^B = 1 \mid (\pi,\sigma)\in B_k).$$

Applying Lemma 1 to equation 1.10, we see

$$\prod_{j=0}^{2n-k-1}(i-j) = i^{2n-k}\exp\left(-\frac{(2n-k)(2n-k-1)}{2i}\right)\left(1+O\left(\frac{n^3}{i^2}\right)\right),$$
$$\prod_{j=0}^{2n-k-1}(n^2-j) = (n^2)^{2n-k}\exp\left(-\frac{(2n-k)(2n-k-1)}{2n^2}\right)\left(1+O\left(\frac{1}{n}\right)\right),$$

and thus setting $2n-k-1\sim 2n-k$ in the above, we get

$$(1.12)\qquad P(x_\pi^B\cdot x_\sigma^B = 1 \mid (\pi,\sigma)\in B_k)$$
$$= (i/n^2)^{2n-k}\exp(-2(1-k/2n)^2(n^2/i-1))(1+O(n^{-2\epsilon})).$$

Let us now break up the sum in (1.11) to

$$(1.13)\qquad \left[\sum_{k=0}^{k_1}+\sum_{k_1+1}^{n}\right](\#B_k)\cdot P(x_\pi^B\cdot x_\sigma^B = 1 \mid (\pi,\sigma)\in B_k)$$

with $k_1=[n^{5/8}]$. For $k\leqq [n^{5/8}]$, (1.12) may be simplified to

$$(1.14)\qquad P(x_\pi^B\cdot x_\sigma^B = 1 \mid (\pi,\sigma)\in B_k)$$
$$=(i/n^2)^{2n-k}\exp(-2(1-k/n)^2(n^2/i-1))(1+O(n^{-1/4-\epsilon})+O(n^{-2\epsilon})).$$

Using equation (1.9) to estimate $\#B_k$, the first sum of (1.13) is given by

$$\sum_{k=0}^{k_1} \frac{1}{k!}(n!)^2 e^{-1}\left(1+O\left(\frac{1}{(n-k)!}\right)\right)\left(\frac{i}{n^2}\right)^{2n-k}$$

$$\cdot\exp\left(-2\left(1-\frac{k}{n}\right)^2\left(\frac{n^2}{i}-1\right)\right)(1+O(n^{-1/4-\epsilon})+O(n^{-2\epsilon}))$$

$$=(n!)^2\exp\left(1-\frac{2n^2}{i}\right)\left(\frac{i}{n^2}\right)^{2n}\left[\sum_{k=0}^{k_1}\frac{1}{k!}\left(\frac{n^2}{i}\exp\left(4\left(\frac{n}{i}-\frac{1}{n}\right)\right)\right)^k\right]$$

$$\cdot(1+O(n^{-1/4-\epsilon})+O(n^{-2\epsilon}))$$

$$=(n!)^2\exp\left(1-\frac{2n^2}{i}\right)\left(\frac{i}{n^2}\right)^{2n}\left[\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{n^2}{i}\exp\left(4\left(\frac{n}{i}-\frac{1}{n}\right)\right)\right)^k\right.$$

$$\left.+O(n^{-((1/8)n^{5/8})})\right]$$

$$\cdot(1+O(n^{-1/4-\epsilon})+O(n^{-2\epsilon}))$$

$$=(n!)^2\exp\left(1-\frac{2n^2}{i}\right)\left(\frac{i}{n^2}\right)^{2n}\left[\exp\left[\frac{n^2}{i}\exp\left(4\left(\frac{n}{i}-\frac{1}{n}\right)\right)\right]\right]$$

$$\cdot(1+O(n^{-1/4-\epsilon})+O(n^{-2\epsilon})).$$

$$(1.15)\qquad =(n!)^2\left(\frac{i}{n^2}\right)^{2n}\exp\left(-\left(\frac{n^2}{i}-1\right)\right)(1+O(n^{-1/4-\epsilon})+O(n^{-2\epsilon})).$$

Using the fact that $\#B_k \leq (1/k!)(n!)^2$, we may bound the second sum of (1.13) by

$$\sum_{k_1+1}^{n}\frac{1}{k!}(n!)^2\left(\frac{i}{n^2}\right)^{2n-k} = (n!)^2\left(\frac{i}{n^2}\right)^{2n} O(n^{-((1/8)n^{5/8})})$$

which is exponentially smaller in n than the contribution of the first sum. This concludes the proof of Theorem II, since the expression (1.15) is the quantity desired.

COROLLARY. *Let $n=1, 2, \cdots$ and $i(n)$ be an integer valued function of n such that $i(n) > n^{3/2+\epsilon}$. Then almost all matrices in $M^n(i(n))$, $n \in \{1, 2, \cdots\}$, have permanent asymptotic to* $\text{Average}_{B\in M^n(i(n))}\,\text{Perm}\,B$.

PROOF. Immediate from Theorems I and II which give the first and second moments of Perm B, $B \in M^n(i)$.

REMARK. A similar technique to the one used above will give the first and second moments of the number of hamiltonian circuits in a random graph with i links and n vertices, $i < n^{3/2-\epsilon}$.

REMARK. If M^n is the class of all $n \times n$ (0, 1)-matrices, then a simple argument of the type used above shows

$$\operatorname*{Average}_{B \in M^n} \text{Perm } B = n! 2^{-n}(1 + O(n^{-1}))$$

and

$$\operatorname*{Average}_{B \in M^n} (\text{Perm } B)^2 = (n!)^2 2^{-2n} e(1 + O(n^{-1})).$$

This result is somewhat surprising since the second moment is not asymptotic to the square of the first moment. It may be checked, however, by the equations

$$\operatorname*{Average}_{B \in M^n} \text{Perm } B = \sum_{i=n}^{n^2} \operatorname*{Average}_{B \in M^n(i)} \text{Perm } B \cdot \# M^n(i)/2^{n^2}$$

and

$$\operatorname*{Average}_{B \in M^n} (\text{Perm } B) = \sum_{i=n}^{n^2} \text{Average } (\text{Perm } B)^2 \cdot \# M^n(i)/2^{n^2}.$$

These sums may be evaluated by the method of steepest descent, and the results check with those above.

REMARK. Further terms of the asymptotic series, with error bounds, may be found for $\text{Average}_{B \in M^n(i)}$ Perm B by proving Lemma 1 again with longer Taylor expansions of the quantity $\log (1 - j/i)$. Evaluation of further terms in the quantity $\text{Average}_{B \in M^n(i)}$ (Perm $B)^2$ is a much harder problem however. It might be possible to go to further terms by evaluating the sum suggested by (1.11) by the method of steepest descent.

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