RADON MEASURES ON GROUPS

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Let G be a Hausdorff topological group with a nontrivial mobile real valued Radon measure. Then G is locally compact. In particular if there is a nontrivial translation invariant Radon measure on G, then G is locally compact.

Let G be a Hausdorff topological group. We shall be dealing with positive and real valued Radon measures [**3**] on G. Following [**1**], a real valued Radon measure μ on G is said to be mobile if, for every compact set $K \subset G$, the function $\sigma \to \mu(\sigma K)$ is continuous on G. In the case of a locally compact Hausdorff group any bounded mobile Radon measure is the indefinite integral (with respect to Haar measure) of a function in $L^1(G)$ [**1**].

We shall prove in this note that if there exists a (nontrivial) mobile real (not necessarily bounded) Radon measure on G, then the group is locally compact (Theorem 2). An immediate consequence is the following converse to the existence of Haar measure. If in a Hausdorff topological group there is a nontrivial ($\not\equiv 0$) left translation invariant Radon measure then the group is locally compact.

Let us first recall

DEFINITION 1. A Radon measure μ on a Hausdorff topological space X is a positive measure defined on the Borel subsets of X satisfying (1) μ is locally finite and (2) μ is inner regular i.e. for every Borel set B, $\mu(B) = \sup\{\mu(K) : \text{compact } K \subset B\}$. A real valued Radon measure ν is a signed Borel measure such that ν^+ and ν^- are both Radon (equivalently ν is the difference of two positive Radon measures, one of which (at least) is totally finite).

We shall need the following fact about Radon measures. For every compact set K and any $\epsilon > 0$, there exists an open set $V \supset K$ such that $\mu(V) < \mu(K) + \epsilon$.

LEMMA 1. *Let μ be a (nonnegative) mobile Radon measure on G. Then, for each compact subset K of G, $\mu(K \triangle \sigma K)$ tends to zero as σ tends to the identity element e.*

PROOF. Given $\epsilon > 0$, let U be an open set containing K such that $\mu(U) < \mu(K) + \epsilon/3$. Since μ is mobile, we can choose a neighbourhood V of e such that $|\mu(K) - \mu(\sigma K)| < \epsilon/3$ for every $\sigma \in V$ and further we can assume that $VK \subset U$. Then, $\mu(U - \sigma K) \leqq 2\epsilon/3$ and we get

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$$\mu(\sigma K \triangle K) \leqq \mu(U - K) + \mu(U - \sigma K) < \epsilon$$

for every $\sigma \in V$. The proof is complete.

THEOREM 1. *If μ is a mobile real Radon measure on G, then both μ^+ and μ^- are mobile.*

PROOF. It can be easily proved that μ^+ would be mobile if for every compact set $K \subset G$, the function $\sigma \rightarrow \mu^+(\sigma K)$ is continuous at e. We may also assume that $\mu^+(G) < +\infty$. Let $G = P \cup N$, with P a positive set for μ and N a negative set for μ. (P and N can be chosen to be Borel sets.) Let $\epsilon > 0$. We choose a compact set $E \subset P$ such that $\mu(P) - \epsilon < \mu(E)$. Since $\mu^-(E) = 0$, we can choose U open $\supset E$ satisfying $\mu^-(U) < \epsilon$. Then, we also have $\mu^+(U - E) < \epsilon$. Now, we select a neighbourhood V of e such that

(i) $VE \subset U$,
(ii) $|\mu(\sigma E) - \mu(E)| < \epsilon$ for each $\sigma \in V$, and
(iii) $|\mu(K \cap E) - \mu[\sigma(K \cap E)]| < \epsilon$ for each $\sigma \in V$.

Then

$$\left| \mu(E) - \mu(\sigma E \cap E) - \mu[\sigma E \cap (U - E)] \right| < \epsilon,$$

and so,

$$\begin{aligned} \left| \mu(E) - \mu(\sigma E \cap E) \right| &\leqq (|\mu|)[\sigma E \cap (U - E)] + \epsilon \\ &\leqq |\mu| (U - E) + \epsilon \\ &< 3\epsilon. \end{aligned} \tag{1}$$

Also,

$$\begin{aligned} \mu[\sigma(K \cap E)] &= \mu(\sigma K \cap \sigma E) \\ &= \mu(\sigma K \cap \sigma E \cap E) + \mu[\sigma K \cap \sigma E \cap (U - E)]. \end{aligned}$$

Hence,

$$\begin{aligned} |\mu(K \cap E) - \mu(\sigma K \cap \sigma E \cap E)| &\leqq |\mu| [\sigma K \cap \sigma E \cap (U - E)] + \epsilon \\ &< 3\epsilon. \end{aligned} \tag{2}$$

Further, since μ is a positive measure on E,

$$\begin{aligned} 0 &\leqq \mu(\sigma K \cap E) - \mu(\sigma K \cap \sigma E \cap E) = \mu[(E \cap \sigma K) - \sigma E] \\ &\leqq \mu(E - \sigma E) = \mu(E) - \mu(E \cap \sigma E) < 3\epsilon \qquad \text{from (1).} \end{aligned} \tag{3}$$

Combining (2) and (3) we get,

$$\left| \mu(K \cap E) - \mu(\sigma K \cap E) \right| < 6\epsilon.$$

Now, for any $\sigma \in V$,

$$\begin{aligned} &| \mu^+(\sigma K) - \mu^+(K) | \\ &\quad = | \mu(\sigma K \cap E) - \mu(K \cap E) + \mu[\sigma K \cap (P-E)] - \mu[K \cap (P-E)] | \\ &\quad \leqq | \mu(\sigma K \cap E) - \mu(K \cap E) | + | \mu[\sigma K \cap (P-E)] | + | \mu[K \cap (P-E)] | \\ &\quad < 8\epsilon. \end{aligned}$$

Since this is valid for every $\epsilon > 0$, we conclude that μ^+ is mobile. Also $\mu^- = \mu^+ - \mu$ is mobile and this completes the proof.

LEMMA 2. *Let μ be a nontrivial Radon measure on G. If there exist K compact $\subset G$, a positive number δ and a neighbourhood V of e satisfying* (i) $0 < \delta < \mu(K)$, *and* (ii) $\forall \sigma \in V$, $\mu(\sigma K \triangle K) < \delta$, *then G is locally compact.*

PROOF. We observe that

$$KK^{-1} \supset \{\sigma : \sigma K \cap K \neq \emptyset\} \supset \{\sigma : \mu(\sigma K \triangle K) < \delta\}.$$

Hence, the compact set KK^{-1} contains the neighbourhood V of e (by hypothesis). It follows that G is locally compact completing the proof.

THEOREM 2. *Let G be a Hausdorff topological group and μ a nontrivial real valued Radon measure on G which is mobile. Then G is locally compact.*

PROOF. Since μ^+ and μ^- are both mobile Radon measures (Theorem 1), we may assume that μ^+ is nontrivial. Let K be a compact set with $\mu^+(K) > 0$ and $0 < \delta < \mu^+(K)$. Then a neighbourhood V of e could be chosen (Lemma 1) so that μ^+, K, δ and V satisfy the conditions of Lemma 2. The required result follows.

The following corollary is an immediate consequence.

COROLLARY 1. *Let G be a Hausdorff topological group with a nontrivial left translation invariant Radon measure. Then G is locally compact.*

COROLLARY 2. *Let G be a topological group of which the underlying space is Souslin. If there exists a nontrivial locally finite mobile (i.e. $\sigma \to \mu(\sigma K)$ continuous) Borel measure on G, then G is locally compact.*

This is a consequence of the fact [**3**, Chapter II] that every locally finite Borel measure on a Souslin space is Radon. This result includes as a particular case a result due to J. C. Oxtoby [**2**, Theorem 1] on polish groups.

Bibliography

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