ON $l$-$l$ SUMMABILITY

B. WOOD

1. Introduction. Let $(E, p_i)$ and $(F, q_j)$ be Fréchet spaces, i.e., locally convex Hausdorff spaces which are metrisable and complete, whose topologies are generated, respectively, by the countable collections $\{p_i\}$ and $\{q_j\}$ of seminorms. Let the infinite matrix $A = (A_{nk})$ consist of entries $A_{nk}$ each of which is a continuous linear operator of $E$ into $F$. Given a sequence $\{x_k\}$ in $E$ we (formally) define a sequence $\{y_n\}$ by

$$y_n = \sum_{k=0}^{\infty} A_{nk}x_k, \quad n = 0, 1, 2, \ldots.$$  

We say the matrix $A$ is an $l$-$l$ method if each series (1.1) converges in $(F, q_j)$ and

$$\sum_{n=0}^{\infty} q_j(y_n) < +\infty, \quad j = 1, 2, \ldots,$$

whenever

$$\sum_{k=0}^{\infty} p_i(x_k) < +\infty, \quad i = 1, 2, \ldots.$$  

We say the method $A$ is absolutely $L$-regular if in addition $\sum_{n=0}^{\infty} y_n = \sum_{k=0}^{\infty} L(x_k)$ whenever $\sum_{k=0}^{\infty} p_i(x_k) < +\infty, \ i = 1, 2, \ldots$. Here $L$ is a prescribed continuous linear operator of $E$ into $F$. It is the purpose of this note to establish necessary and sufficient conditions which ensure that $A$ be $l$-$l$ or absolutely $L$-regular. For the classical case $(E, F$ the complex numbers with the usual topology) these conditions were given by Mears [3] and Knopp and Lorentz [1] and for the Banach space setting by Lorentz and Macphail [2].

2. Theorems.

Theorem 2.1. The matrix $A = (A_{nk})$ defining series to series transformations from the $F$-space $(E, p_i)$ into the $F$-space $(F, q_j)$ is $l$-$l$ if and only if

$$(2.1) \text{for each bounded set } M_a \text{ in } E \text{ and for each fixed } j,$$

$$\sum_{n=0}^{m} q_j(A_{nv}(x_v)) \leq K_{a,j} \quad \text{for } m, v = 0, 1, 2, \ldots$$

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and $x_v \in M_\alpha$, $v = 0, 1, 2, \cdots$.

The proof of Theorem 2.1 requires the following lemmas. Lemma 2.2 is known [4], while Lemma 2.3 is a minor modification of Lemma 2.4 of [5].

**Lemma 2.2.** If $E$ and $F$ are locally convex spaces and $E$ is quasicomplete then any collection of continuous linear operators from $E$ into $F$ which is simply bounded is bounded for the topology of uniform convergence on bounded sets.

**Lemma 2.3.** If $\sum_{k=0}^{\infty} A_{nk} x_k$ converges in $F$ for every sequence $\{x_k\}$ in $E$ such that $\sum_{k=0}^{\infty} p_i(x_k)$ converges ($i = 1, 2, \cdots$), then the sequence $\{A_{nk}\}$, $k = 0, 1, \cdots$, of continuous linear operators from $E$ into $F$ is bounded (for fixed $n$) for the topology of uniform convergence on bounded sets.

**Proof of Theorem 2.1.** Assume $A = (A_{nk})$ is $l$-$l$ and consider the linear space $E_1$ of sequences $\{x_k\}$ in $E$ such that $\sum_{k=0}^{\infty} p_i(x_k) < + \infty$ ($i = 1, 2, \cdots$). For $x = \{x_k\}$ in $E_1$ define $P_i(x) = \sum_{k=0}^{\infty} p_i(x_k)$. Then, for each $i$, $P_i$ is a seminorm on $E_1$ and the locally convex space $(E_1, P_i)$ is complete. Now, since $A = (A_{nk})$ is $l$-$l$, each series $\sum_{k=0}^{\infty} A_{nk}(x_k)$ converges in $(F, q_j)$ whenever $\sum_{k=0}^{\infty} p_i(x_k) < + \infty$, $\{x_k\}$ in $E$. It follows from Lemma 2.3 that $\{A_{nk}\}$, $k = 0, 1, \cdots$, is bounded for the topology of uniform convergence on bounded sets. We shall show that this implies

(2.2) for each $n = 0, 1, 2, \cdots$, $i = 1, 2, \cdots$ and $j = 1, 2, \cdots$, there exists a number $K_{n,j,i} \geq 0$ such that $q_j(A_{nk}(x)) \leq K_{n,j,i}p_i(x)$, $x \in E$, $k = 0, 1, 2, \cdots$.

For each $k = 0, 1, \cdots$ and fixed $i$, $j$, $n$, define $\mu_k(x) = q_j(A_{nk}(x))$, $x \in E$. Then $\mu_k$ is a seminorm on $E$. It follows from the fact that $\{A_{nk}\}$ ($k = 0, 1, \cdots$) is bounded for the topology of uniform convergence on bounded sets that there exists a number, $K_{n,j,i} \geq 0$, such that $p_i(x) < 1$ and $k = 0, 1, \cdots$ imply $\mu_k(x) \leq K_{n,j,i}$. If $K_{n,j,i} > 0$ it is easy to see that (2.2) holds. On the other hand, if $K_{n,j,i} = 0$ (2.2) follows from elementary properties of seminorms (see, e.g., the proof of Theorem 2.1 in [5]). Thus for each fixed $n = 0, 1, \cdots$ the linear operator $T_n$ defined by $T_n(x) = \sum_{k=0}^{\infty} A_{nk}(x_k)$, $x = \{x_k\} \in E_1$, is in $L(E_1, F)$, i.e., $T_n$ is a continuous linear operator from $E_1$ into $F$. Let $F_1$ denote the linear space of sequences $\{y_k\}$ in $F$ such that $\sum_{k=0}^{\infty} q_j(x_k) < + \infty$. For $y = \{y_k\} \in F_1$ define $Q_j(y) = \sum_{k=0}^{\infty} q_j(y_k)$. Then $(F_1, Q_j)$ is a locally convex complete seminormed space. Define the operators $U_m$, $m = 0, 1, \cdots$, by $U_m(x) = \{y_n\}$ where
\[ y_n = T_n(x), \quad n = 0, 1, \ldots, m, \]
\[ = 0, \quad n > m, \]
and \( x \in E_1 \). Thus \( U_m \in L(E_1, F_1), \ m = 0, 1, \ldots \). Since \( A \) is \( l-l \), \( \{ U_m(x) \} \) converges in \( (F_1, Q_j) \) whenever \( x \in E_1 \). It now follows from Lemma 2.2 that \( U_m, \ m = 0, 1, \ldots \), is bounded for bounded convergence on \( L(E_1, F_1) \). Therefore, for each fixed \( j \) and each bounded set \( M \) in \( E_1 \)
\[
\sup_{x \in M} Q_j(U_m(x)) \leq K_{M,j}, \quad m = 0, 1, \ldots.
\]
Consider a bounded set \( M_a \) in \( E \). Say \( M_a \) consists of points \( x \) such that \( p_i(x) < \alpha_i \). Consider sequences of the form \( \{x_0, 0, 0, \ldots \} \), \( \{0, x_1, 0, \ldots \} \), \( \{0, 0, x_20, \ldots \} \) \ldots, where the \( x_i \)'s are in \( M_a \). All such sequences are in the same bounded set \( U_a \) of \( (E_1, P_i) \). Therefore,
\[
Q_j(U_m(x)) = Q_j(\{T_0(x), T_1(x), \ldots, T_m(x), 0, 0, 0, \ldots\})
\]
\[
= \sum_{n=0}^{m} q_j(T_n(x)) = \sum_{n=0}^{m} q_j(A_{nv}(x_v))
\]
\[
\leq K_{m_a,j} = K_{a,j},
\]
for \( m, v = 0, 1, 2, \ldots \) and \( x_v \in M_a \) for each \( v \), i.e., (2.1) holds.

Conversely, suppose (2.1) is true. Let \( x = x_k \in E_1 \). Then \( x_k \in (\text{same}) \) bounded set \( M_a \) in \( E \) for \( k = 0, 1, \ldots \). For each \( j \) and each \( n = 0, 1, \ldots \), we claim that
\[
\sum_{k=0}^{\infty} q_j(A_{nk}(x_k)) < +\infty.
\]
For, by (2.1), given \( j \) there exists \( K_{a,j} \geq 0 \) such that \( q_j(A_{nk}(x_k)) \leq K_{a,j}, n, k = 0, 1, \ldots \). It now follows, as in the first part of the proof, that there exists a number \( R = R(n, j, i) \) such that \( q_j(A_{nk}(x_k)) \leq Rp_i(x_k) \) for \( k = 0, 1, \ldots \). Thus (2.3) is valid.

Since
\[
y_m = \sum_{n=0}^{m} q_j \left( \sum_{k=0}^{\infty} A_{nk}(x_k) \right)
\]
is a nondecreasing sequence of nonnegative numbers, it suffices to show that \( \{y_m\} \) is bounded above in order to conclude that \( A \) is \( l-l \). For a given \( j \), \( x \in E \) and \( v = 0, 1, \ldots \), define
\[
S_j(x) = \sum_{n=0}^{\infty} q_j(A_{nv}(x)).
\]
It follows, using (2.1), that $S_j$ is a seminorm on $E$. Property (2.1) now implies, as before, the existence of a number $T \geq 0$, where $T$ depends on $j$, $i$ but is independent of $n$ and $v$ ($n, v = 0, 1, \ldots$), such that $S_j(x) \leq Tp_i(x)$ for $x \in E$. Using (2.3) we obtain easily that $\{y_m\}$ is bounded above and the proof is complete.

Under (2.1) we have

$$\lim_{m} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A_{nk}(x_k) = \lim_{m} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} A_{nk}(x_k) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} A_{nk}(x_k).$$

The following theorem is now obvious.

**Theorem 2.4.** The method $A = (A_{nk})$, considered as an $l-l$ method, is absolutely $L$-regular if and only if (2.1) holds and also

$$\lim_{m} \sum_{n=0}^{m} A_{nk}(x_k) = L(x_k), \quad k = 0, 1, \ldots,$$

for $\{x_k\} \in E_i$.

**References**


**University of Arizona, Tucson, Arizona 85721**