SYMMETRY OF GENERALIZED GROUP ALGEBRAS

KJELD B. LAURSEN

In this note we shall consider the generalized group algebras $B^p(G, A)$, where $G$ is a compact Hausdorff group, $A$ a Banach-* algebra, and $1 \leq p < \infty$. These spaces have been studied by Spicer [6] and [7] and are defined as the spaces of functions $f: G \to A$ for which

$$\left[ \int |f(g)|^p dm(g) \right]^{1/p} < \infty.$$

$B^p(G, A)$ is normed by defining

$$|f|_p = \left[ \int |f(g)|^p dm(g) \right]^{1/p}$$

and involution is defined as usual: $f^*(g) = f(g^{-1})^*$. We prove the following

**Theorem.** If $G$ is a compact group and $A$ is a Banach algebra with (continuous) involution, then $B^p(G, A)$ is symmetric if and only if $A$ is symmetric.

A Banach-* algebra is symmetric if elements $f^* f$ have nonnegative spectrum. This is the case if and only if hermitian elements have real spectra [5].

We first observe that it suffices to show that $A$ is symmetric iff $B^1(G, A)$ is symmetric, because $B^1(G, A)$ symmetric $\Rightarrow B^p(G, A)$ symmetric for any $p$, $1 \leq p < \infty$, $\Rightarrow A$ symmetric $\Rightarrow B^1(G, A)$ symmetric. The proof of the first of these implications will be accomplished by showing that if $t \in B^p(G, A) \subseteq B^1(G, A)$ then the spectrum of $t$ in $B^p(G, A)$, $\sigma_p(t)$ equals the spectrum of $t$ in $B^1(G, A)$, $\sigma_1(t)$. Since $B^p(G, A) \subseteq B^1(G, A)$, clearly $\sigma_1(t) \subseteq \sigma_p(t)$. On the other hand, $B^p(G, A)$ is an ideal in $B^1(G, A)$ [6]. Recall that $0 \neq \lambda \in \sigma_1(t)$ iff $t/\lambda$ has a quasi-inverse, $y_\lambda$, say, in $B^1(G, A)$ [4, p. 28]. $t/\lambda$ and $y_\lambda$ satisfy the relationship

$$t/\lambda + y_\lambda - (t/\lambda) * y_\lambda = 0$$

or

$$y_\lambda = (t/\lambda) * y_\lambda - t/\lambda.$$

Using the fact that $B^p(G, A)$ is an ideal in $B^1(G, A)$ we conclude

Received by the editors June 4, 1969.

318
that \( y_{\lambda} \in B^p(G, A) \) i.e. \( \lambda \in \sigma_p(t) \). If neither \( B^p(G, A) \) nor \( B^1(G, A) \) contains an identity \( 0 \in \sigma_1(t) \) and \( 0 \in \sigma_p(t) \). On the other hand, if \( G \) is a finite group, then obviously \( t \in B^p(G, A) \) has an inverse in \( B^p(G, A) \) if and only if \( t \) has an inverse in \( B^1(G, A) \), because \( B^1(G, A) = B^p(G, A) \) setwise. Consequently, we have shown

**Lemma 1.** \( \sigma_1(t) = \sigma_p(t) \) for all \( t \in B^p(G, A) \) and any \( p, 1 \leq p < \infty \).

From this lemma it follows that if \( B^1(G, A) \) is symmetric and \( t \in B^p(G, A) \), then \( -1 \in \sigma_1(t^*t) \) and therefore \( -1 \in \sigma_p(t^*t) \). This shows that \( -t^*t \) is quasi-regular in \( B^p(G, A) \) for any \( t \in B^p(G, A) \); therefore \( B^p(G, A) \) is symmetric if \( B^1(G, A) \) is symmetric.

Next we prove the second implication. Suppose \( B^p(G, A) \) is symmetric. We show that \( A \) is symmetric. Simply embed \( A \) in \( B^p(G, A) \) by considering the isometric image of \( A \) in \( B^p(G, A) \). This identification shows immediately that if \( B^p(G, A) \) is symmetric, then \( A \) is symmetric.

To show that \( B^1(G, A) \) is symmetric, provided that \( A \) is, is somewhat more complicated. The proof given here depends on the minimal ideal structure of \( L^1(G) \) via the identification \( B^1(G, A) = L^1(G) \otimes_A \) \([6]\), based on a result by Grothendieck \([1, p. 59]\). We present the proof as a sequence of lemmas.

The first two of these are proved in \([7]\).

**Lemma 2.** Let \( X_1 \) be a finite-dimensional Banach space and \( X_2 \) be any Banach space. Let \( \{l_1, \cdots, l_n\} \) be any basis of unit vectors for \( X_1 \); if \( t \in X_1 \otimes X_2 \) then \( t \) has a unique representative \( t = \sum_{i=1}^n l_i \otimes y_i \). Define

\[
\phi: X_1 \otimes X_2 \to \sum \oplus_n X_2 \quad \text{by} \quad \phi(t) = (y_1, \cdots, y_n).
\]

\( \phi \) is an algebraic isomorphism onto.

**Lemma 3.** \( X_1 \) and \( X_2 \) as in Lemma 2. If we norm \( \sum \oplus_n X_2 \) by defining

\[
| (y_1, \cdots, y_n) | = \sum_{i=1}^n | y_i |,
\]

then \( \phi \) as defined above is a homeomorphism of \( \sum \oplus_n X_2 \) and \( X_1 \otimes \gamma X_2 \).

Note that since \( X_1 \) is finite-dimensional the algebraic tensor product normed with the greatest cross norm is complete.

**Lemma 4.** Suppose \( X_1 \) is a simple finite-dimensional annihilator algebra with proper involution (see \([4]\)) and continuous quasi-inversion. Suppose \( X_2 \) is a Banach-\( \ast \)-algebra. Then \( \phi \) defined in Lemma 2 is a \( \ast \)-algebra-isomorphism.
Proof. The assumptions on $X_1$ are made simply to ensure that $X_1$ has a basis $\{e_i\}$ consisting of orthonormal, hermitian idempotents [3, p. 330], i.e. $\{e_i\}$ satisfies
\[ e_i e_j = \delta_{ij} e_j, \quad \text{for all } i, j = 1, \ldots, n, \]
and
\[ e_i^* = e_i, \quad \text{for all } i = 1, \ldots, n. \]
If $t \in X_1 \otimes \gamma X_2$ then as before $t = \sum e_i \otimes y_i$ and $\phi(t) = (y_1, \ldots, y_n)$. Hence $t^* = \sum e_i^* \otimes y_i^* = \sum e_i \otimes y_i^*$ and $\phi(t^*) = (y_1^*, \ldots, y_n^*) = [\phi(t)]^*$. Moreover, if $t_1 = \sum e_i \otimes x_i$ and $t_2 = \sum e_j \otimes y_j$ then
\[ t_1 t_2 = \sum_{ij} e_i e_j \otimes x_i y_j = \sum_i e_i \otimes x_i y_i \]
so that
\[ \phi(t_1 t_2) = (x_1 y_1, \ldots, x_n y_n) = (x_1, \ldots, x_n) (y_1, \ldots, y_n) = \phi(t_1) \phi(t_2). \]

Corollary 1. $X_1$ and $X_2$ as in Lemma 4. $X_1 \otimes \gamma X_2$ is symmetric if and only if $X_2$ is symmetric.

Remark. The assumptions about $X_1$ imply that $X_1$ is symmetric [4, p. 266].

Proof. By Lemma 3 and Lemma 4 it suffices to show that $\sum_n \oplus X_2$ is symmetric iff $X_2$ is symmetric. But this is an immediate consequence of the fact that
\[ \sigma(y_1, \ldots, y_n) = \bigcup_{i=1}^n \sigma(y_i) \]
for any $(y_1, \ldots, y_n) \in \sum_n \oplus X_2$.

We now specialize to $B^1(G, A) = L^1(G) \otimes \gamma A$. The theory to be developed depends on the minimal ideal structure of $L^1(G)$.

If $S \subseteq L^1(G)$ then $(S \otimes A)_\gamma$ will denote the $\gamma$-closure of $S \otimes A$ in $L^1(G) \otimes \gamma A$.

Lemma 5. If $M \subseteq L^1(G)$ is a minimal two-sided closed ideal, then $(M \otimes A)_\gamma$ is a closed ideal in $B^1(G, A)$, symmetric if and only if $A$ is symmetric.

Proof. Since $G$ is compact, $M$ is a finite-dimensional simple annihilator algebra with proper involution and continuous quasi-inversion [3, VI]. Clearly $(M \otimes A)_\gamma$ is a closed ideal; if $t_1 = \sum_{i=1}^n x_i \otimes y_i \in M \otimes A$ and
Let \( t_2 = \sum_{j=1}^{\infty} u_j \otimes v_j \in L^1(G) \otimes_\gamma A \)

then

\[
t_1 t_2 = \sum_{i,j} x_i u_j \otimes y_i v_j \in (M \otimes A)_\gamma.
\]

Similarly \( t_2 t_1 \in (M \otimes A)_\gamma \). It is easy to see that \( M \otimes_\gamma A \) and \( (M \otimes A)_\gamma \) are *-isomorphic, using the technique of Lemma 2. Since \( M \otimes_\gamma A \) is symmetric iff \( A \) is symmetric, the conclusion follows by the above and Lemma 4.

**Lemma 6.** Let \( \{ M_i \}_{i=1}^n \) be minimal two-sided ideals of \( L^1(G) \); if \( A \) is symmetric, then \( \sum_{i=1}^{n} \bigoplus (M_i \otimes A)_\gamma \subset B^1(G, A) \) is symmetric.

**Proof.** If \( A \) is symmetric, then \( (M_i \otimes A)_\gamma \) is symmetric (Lemma 5). The rest follows as in the proof of Corollary 1.

Now we are able to complete the proof of the theorem. Suppose \( A \) is symmetric. We must show that \( B^1(G, A) \) is symmetric; from this it will follow that \( B^n(G, A) \) is symmetric (Lemma 1). In accordance with [5] we show that hermitian elements in \( B^1(G, A) \) have real spectra. Adapting a construction in [2, Theorem (28.53)] to the present situation we can find a net of functions \( \{ u_\alpha \} \) with the following properties:

(i) each \( u_\alpha \) is complex-valued continuous, nonnegative, positive-definite and central.

(ii) \( \int u_\alpha dm = 1 \) for all \( \alpha \).

(iii) \( u_\alpha * f = f * u_\alpha \) for any \( f \in L^1(G) \).

Clearly each \( u_\alpha \) generates an operator \( T_\alpha \) in \( B^1(G, A) \) defined as follows

\[
f = \sum x_i \otimes y_i \in B^1(G, A) = L^1(G) \otimes_\gamma A \Rightarrow \]

\[
T_\alpha f = \sum u_\alpha * x_i \otimes y_i.
\]

Equally clear is it that \( T_\alpha \) approximates the identity of \( B^1(G, A) \), i.e.

\[
\sum u_\alpha * x_i \otimes y_i \to \sum x_i \otimes y_i.
\]

Now, let \( f \) be a hermitian element of \( B^1(G, A) \). We will use the notation \( u_\alpha * f \) for \( T_\alpha f \in B^1(G, A) \). From \( \{ u_\alpha * f \} \) we can pick a sequence \( \{ u_n * f \} \) such that

\[
| u_n * f - f | < 1/n, \quad n = 1, 2, \ldots.
\]

Let \( \mathcal{F} = \{ M \} \) be the collection of minimal two-sided ideals in \( L^1(G) \). Since \( G \) is compact any irreducible representation of \( L^1(G) \) is
realizable as left translation in some $M \subseteq \mathfrak{F}$ [3, p. 434]. Moreover, each $u_n$ being positive definite we have that

$$u_n(\cdot) = \sum_{M \in \mathfrak{F}} c_n(M) \chi_M(\cdot)$$

where $\chi_M$ is the character of $M$, where $c_n(M) \geq 0$ and where $\sum_{M \in \mathfrak{F}} c_n(M) \chi_M(e) < \infty$. Again following [2], for each $u_n$ we can choose a finite partial sum, $u'_n$, of the series for $u_n$ so that $|u'_n - u_n|_\infty < 1/2n$. Setting $u''_n = \frac{1}{2}(u'_n + u'_n^*)$ we get a continuous hermitian central trigonometric polynomial for which $|u''_n - u_n|_\infty < 1/n$. If we normalize $u''_n$ to obtain $v_n$, i.e.

$$v_n = u''_n / |u''_n|_1, \quad n = 1, \ldots,$$

then it is clear that $v_n \ast f = f$. Since $v_n \ast f = f \ast v_n$ each $v_n \ast f$ is hermitian. Also since $v_n \ast f$ and $f$ commute we can make use of [4, (1.6.17)]. Consequently, it suffices to show that $v_n \ast f$ has a real spectrum. But since $v_n$ is a finite linear combination of characters it follows that

$$v_n \ast f \in \sum_{i \in K} \oplus (M_i \otimes A)_\gamma$$

where $K$ is finite set. Lemma 6 then implies that $v_n \ast f$ has real spectrum. This completes the proof.

**Corollary.** Let $G$ be a compact group and $H$ a locally compact group. $L^1(G \times H)$ is symmetric if and only if $L^1(H)$ is symmetric.

**Bibliography**


University of Aarhus, Aarhus, Denmark