NOTE ON HILLE'S EXPONENTIAL FORMULA

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ABSTRACT. In an earlier paper of the author it was shown that we would not be able to obtain a better estimate for Hille's first exponential formula than $Kw_B(\tau^{1/2}, T(\cdot)f)$, where $w_B(\delta, T(\cdot)f)$ is the global modulus of continuity of $T(t)f$, $t\in[0, B]$. It is shown in this paper that this estimate can actually be achieved.

Let $T(t)$ be a strongly continuous semigroup of operators on a Banach space into itself, and let $W_L(\delta; T(\cdot)f)$ be the rectified modulus of continuity in $[0, L]$ given by:

$$(1)\quad W_L(\delta; T(\cdot)f) = \{\sup\|T(t) - T(s)\|, \ |t-s| < \delta, 0 \leqq t, s < L\}.$$

Trying to answer a question mentioned by P. Butzer and H. Berens [1] the author [2] estimated

$$(2)\qquad \|\exp(tA_\tau)f - T(t)\| \leqq W_L(\tau^\gamma; T(\cdot)f) + K\tau^\gamma\|f\|$$

for all $\gamma<1/2$, where $A_\tau=\tau^{-1}(T(\tau)-I)$, and showed that (2) is false for $\gamma>1/2$. It remained to find if an estimation similar to (2) is valid for $\gamma=1/2$. This will be answered here positively as follows:

THEOREM 1. *Let $\{T(t); 0\leqq t<\infty\}$ be a strongly continuous semigroup on a Banach space B into itself. Then*

$$(3)\quad \|\exp(tA_\tau)f - T(t)f\| \leqq (L^{1/2}+1)W_L(\tau^{1/2}, T(\cdot)f) + K\tau^{1/2}\|f\|,$$

where K is independent of τ, $t\leqq L-\delta$ and $\tau^\gamma<\delta$ for some $\gamma<1/2$.

PROOF. Following the proof of Theorem 2.1 in [1] we write

$$(4)\qquad \begin{aligned}\|\exp(tA_\tau)f - T(t)f\| &\leqq e^{-t/\tau}\sum_{n=0}^{\infty}(t/\tau)^k(1/k!)\|T(k\tau)f - T(t)f\| \\ &\equiv \Sigma' + \Sigma'',\end{aligned}$$

where Σ' and Σ'' are the sums on all $k\geqq 0$ satisfying $|k-(t/\tau)|\leqq\tau^{\gamma-1}$ and $|k-(t/\tau)|>\tau^{\gamma-1}$, respectively, for some fixed γ, $0<\gamma<1/2$. For Σ'' we use the estimate given in [2]. We recall that for λ, $\eta\geqq 0$, $W_L(\lambda\cdot\eta, T(\cdot)f)\leqq(\lambda+1)W_L(\eta, T(\cdot)f)$. Substituting $\eta=\tau^{1/2}$ we have

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$$\Sigma' \leqq e^{-t/\tau} \sum_{|k-t/\tau| \leq \tau^{-1}} \left(\frac{t}{\tau}\right)^k \frac{1}{k!} W_L(|k\tau - t|, T(\cdot)f)$$

$$\leqq e^{-t/\tau} \sum_{k=0}^{\infty} \left(\frac{t}{\tau}\right)^k \frac{1}{k!} \left(\frac{|k\tau - t|}{\tau^{1/2}} + 1\right) W_L(\tau^{1/2}, T(\cdot)f).$$

Using now Lemma 1.2.1 of [1, pp. 18–19] and especially $\sum_{k=0}^{\infty} |k-u| u^k/k! \leqq \sqrt{u}\, e^u$, and recalling that $t<L$, we obtain

$$\Sigma' \leqq (L^{1/2} + 1) W_L(\tau^{1/2}; T(\cdot)f). \qquad \text{Q.E.D.}$$

REMARK 1. Under assumptions of Theorem 1 we have also

$$\|\exp(tA_\tau)f - T(t)f\| \leqq K W_L(\tau^{1/2}, T(\cdot)f) \tag{5}$$

where K does not depend on τ.

REMARK 2. It was shown in [2] that the result above is best possible up to a constant.

REFERENCES

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