ON THE INVERSE OF AN INTEGRAL OPERATOR

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We wish to consider the integral equation

\[ f(x) = \frac{i}{2} \int_{-1}^{1} H_0^{(1)}(k | x - t |) \phi(t) dt. \]

Here \( H_0^{(1)} \) denotes the zero order Hankel function of the first kind. \( k \) is a nonzero constant with \( \text{Re } k \geq 0, \text{Im } k \geq 0 \). Recall that for small \( r \) we have

\[ \frac{i}{2} H_0^{(1)}(kr) = \frac{1}{\pi} \log \frac{1}{r} + h(r) \]

where \( h(r) \) and \( h'(r) \) are finite at \( r = 0 \). The equation (1) arises in connection with the solution of the reduced wave equation in the plane slit along the \( x \)-axis from \(-1 \) to \(+1 \) [1].

In [1] the following result was proven: Let \( h \) denote the class of complex functions \( \phi \) which are Hölder continuous in a neighborhood of each point of \((-1, 1)\) and further satisfy the condition that near \( x = 1, |\phi(x)| \leq \kappa/(1-x)^\alpha, 0 \leq \alpha < 1 \) and near \( x = -1, |\phi(x)| \leq \kappa/(1+x)^\alpha \). Then given \( f(x) \) such that \( f' \) is Hölder continuous, equation (1) has a unique solution, \( \phi \in h \). In this paper we will consider equation (1) as a mapping from one Hilbert space into another. We will show that if the domain and range spaces are defined appropriately the integral operator in (1) becomes a one-to-one continuous mapping of one Hilbert space onto another and hence by Banach’s open mapping theorem has a continuous inverse. It will be shown that if \( f \) is sufficiently smooth, the solutions found here coincide with those found in [1].

Let \( p(t) = (1-t^2)^{-1/2}, -1 < t < 1 \) and \( q(t) = (1-t^2)^{1/2} = 1/p(t), -1 < t < 1 \). We define three spaces:

\[ L_2(p) = \left\{ f \left| \int_{-1}^{1} |f|^2(1-t^2)^{-1/2} dt < \infty \right. \right\}; \]

\[ L_2(q) = \left\{ f \left| \int_{-1}^{1} |f|^2(1-t^2)^{1/2} dt < \infty \right. \right\}; \]

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$W_2^1(q) = \{ f \mid f$ is absolutely continuous on $[-1, 1]$ and $f'$ (which exists a.e. with respect to Lebesgue measure) $\in L_2(q) \}.$

If in $L_2(p)$ we define $\|f\|_{L_2(p)}^2 = \int_{-1}^{1} |f(t)|^2 (1-t^2)^{-1/2} \, dt$ and in $L_2(q)$ we define $\|f\|_{L_2(q)}^2 = \int_{-1}^{1} |f(t)|^2 (1-t^2)^{1/2} \, dt$ then these spaces are Hilbert spaces. In $W_2^1(q)$ we define

$$\|f\|_{W_2^1(q)}^2 = \|f\|_{L_2(q)}^2 + \|f'\|_{L_2(q)}^2.$$ 

We then have

**Theorem 1.** Under the above norm $W_2^1(q)$ is a Hilbert space.

**Proof.** We first note that $L_2(q) \subseteq L_1(-1, 1)$ (the usual class of functions integrable over $(-1, 1)$ with respect to Lebesgue measure) and the injection is continuous. This can be seen by using the Schwarz inequality in $L_2(q)$.

Now suppose $\{f_n\}$ is a Cauchy sequence in $W_2^1(q)$. In particular $\{f_n\}$ is Cauchy in $L_2(q)$. Thus there exists $g \in L_2(q)$ such that $\|f_n - g\|_{L_2(q)} \to 0$. By the above remark $f'_n, g \in L_1(-1, 1)$. Thus

$$f_n(x) = f_n(-1) + \int_{-1}^{x} f'_n(t) \, dt.$$ 

Hence

$$f_n(-1) - f_m(-1) = f_n(x) - f_m(x) - \int_{-1}^{x} (f'_n(t) - f'_m(t)) \, dt.$$ 

Thus

$$\left| f_n(-1) - f_m(-1) \right|^2 \leq 2 \left| f_n(x) - f_m(x) \right|^2 + 2 \|f'_n - f'_m\|_{L_2(q)}^2.$$ 

If we multiply by $g(t)$ and integrate from $-1$ to $1$ we find

$$\frac{\pi}{2} \left| f_n(-1) - f_m(-1) \right|^2 \leq 2 \|f_n - f_m\|_{L_2(q)}^2 + \pi \|f'_n - f'_m\|_{L_2(q)}^2.$$ 

Thus

$$\left| f_n(-1) - f_m(-1) \right|^2 \leq \frac{4}{\pi} \|f_n - f_m\|_{L_2(q)}^2 + 2\pi \|f'_n - f'_m\|_{L_2(q)}^2 \to 0$$ 

as $m, n \to \infty$. Thus $f_n(-1) \to C$ as $n \to \infty$. Let $f(x) = C + \int_{-1}^{1} g(t) \, dt$. Then $f$ is absolutely continuous and
\[ f(x) - f_n(x) = C - f_n(-1) + \int_{-1}^{x} (g(t) - f'_n(t)) \, dt. \]

Thus
\[ |f(x) - f_n(x)|^2 \leq 2 |C - f_n(-1)|^2 + 2 \| g - f'_n \|^2. \]

As above it then follows that
\[ \| f - f_n \|^2_{L^2(\Omega)} \leq \pi |C - f_n(-1)|^2 + 2\pi \| g - f'_n \|^2_{L^2(\Omega)} \to 0 \]
as \( n \to \infty \). Thus \( \| f_n - f \|^2_{W^1_2(\Omega)} \to 0 \) as \( n \to \infty \).

We now consider the operator defined by (1). Let

\[ \psi(x) = \frac{i}{2} \int_{-1}^{1} H_0^{(1)}(k \, |x - t|) \phi(t) \, dt = (L\phi)(x). \]

As is pointed out in [1] if \( \phi \) is Holder continuous we may differentiate under the integral sign and obtain (in view of (2)):

\[ \psi'(x) = \frac{1}{\pi} \int_{-1}^{1} \phi(t) \frac{dt}{x - t} + \int_{-1}^{1} k(t, x) \phi(t) \, dt \]

where the first term must be taken as a Cauchy Principal Value and in the second term \( k(t, x) \) is a continuous kernel.

We now consider (4) as an equation in \( L^2(\Omega) \). Let \( F: L^2(\Omega) \to L^2(p) \) be defined by \((F\phi)(t) = (1 - t^2)^{1/2} \phi(t)\). Then \( F \) is an isometry of \( L^2(\Omega) \) onto \( L^2(p) \). Define an operator \( T \) by

\[ Tg = \frac{1}{\pi} \int_{-1}^{1} \frac{g(t)}{x - t} \frac{1}{(1 - t^2)^{1/2}} \, dt. \]

Then we have the following theorem [2].

**Theorem 2.** The operator defined by (5) is a continuous mapping from \( L^2(p) \) onto \( L^2(\Omega) \). Its null space is one dimensional and is spanned by the function \( g(x) \equiv 1 \). Further the restriction, \( T_0 \), of \( T \) to the orthogonal complement \( H(p) \) of this null space is an isometry of \( H(p) \) onto \( L^2(\Omega) \) with inverse mapping

\[ T_0^{-1} h = \frac{1}{\pi} \int_{-1}^{1} \frac{h(t)}{t - x} (1 - t^2)^{1/2} \, dt. \]

Thus the mapping \( \pi^{-1} \int_{-1}^{1} \phi(t)/(x - t) \, dt \) can be written as \( TF\phi \). We see that it maps \( L^2(\Omega) \) continuously onto \( L^2(\Omega) \) with a one dimensional null space spanned by \( p(t) = (1 - t^2)^{-1/2} \). We recall the definition of
the index of an operator $S$ from one linear space $X$ to another linear space $Y$. Suppose $S$ has a finite dimensional null space $N(S)$, $\dim N(S) = \alpha(S)$, and that the range of $S$, $R(S)$, has finite codimension. Then $\text{codim } R(S) = \dim Y/R(S) = \beta(S)$. (In which case $S$ is said to be a Fredholm operator.) The integer $i(S) = \alpha(S) - \beta(S)$ is called the index of the operator $S$. Thus we have that $TF$ is a Fredholm operator with $\alpha(TF) = 1$, $\beta(TF) = 0$. Thus $i(TF) = 1$. Since $k(t, x)$ is continuous so that

$$\int_{-1}^{1} \int_{-1}^{1} |k(t, x)|^2(1-t^2)^{-1/2}(1-x^2)^{1/2} dx dt < \infty,$$

$\int_{-1}^{1} k(t, x)\phi(t)dt$ represents a compact operator, $K_0$, from $L_2(q)$ into $L_2(q)$. Now the operator $TF$ admits a left regularization [3], i.e. there exists a linear bounded operator $Q$ mapping $L_2(q)$ into $L_2(q)$ such that $Q(TF) = I + K$ where $I$ is the identity in $L_2(q)$ and $K$ is a compact operator (we take $Q = F^{-1} T_0^{-1}$). Then $K = -P_0$ where $P_0$ is the projection onto the space spanned by $\phi(t) = 1/(1-t^2)^{1/2}$. We then note:

**Theorem 3 [3].** If a bounded operator $A$ admits a left regularization and has finite index and $K$ is any compact operator we have $i(A + K) = i(A)$.

Hence we conclude that the mapping defined by the right-hand side of (4) is a continuous mapping of $L_2(q)$ into $L_2(q)$ with index equal to 1.

We return now to the operator $L$ defined by (3). We have

$$\int_{-1}^{1} \int_{-1}^{1} |H_0^{(1)}(k|x-t|)|^2(1-t^2)^{-1/2}(1-x^2)^{1/2} dx dt < \infty.$$  

Thus $L$ is a continuous (compact) operator from $L_2(q)$ into $L_2(q)$.

**Theorem 4.** The operator $L$ maps $L_2(q)$ into $W^1_2(q)$.

**Proof.** Given $\phi \in L_2(q)$. Let

$$\psi = L\phi, \quad \chi = TF\phi + K_0\phi.$$  

Let $\{\phi_n\}$ be a sequence of Hölder continuous functions $\exists \|\phi_n - \phi\|_{L_2(q)} \to 0$. Let $\psi_n = L\phi_n$.

Then we know that $\psi_n$ is differentiable on $(-1, 1)$ and $\psi'_n = TF\phi_n + K_0\phi_n$. By continuity of the mappings $L$ and $TF + K_0$ we see that $\{\psi_n\}$ and $\{\psi'_n\}$ are Cauchy sequences in $L_2(q)$ i.e. $\{\psi_n\}$ is a Cauchy sequence in $W^1_2(q)$. By Theorem 1 $\exists \psi_0 \in W^1_2(q)$ $\exists \|\psi_n - \psi_0\|_{W^1_2(q)} \to 0$. Hence $\|\psi_n - \psi_0\|_{L_2(q)} \to 0$ but $\psi_n \to \psi$ in $L_2(q)$. Thus $\psi = \psi_0$ a.e. In fact $\psi = \psi_0$ since $\psi$ can easily be shown to be continuous and $\psi_0$ is absolutely continuous. Also $\chi = \psi'_0$ a.e. Hence the theorem is proven.
Theorem 5. The operator $L$ is a one-to-one map of $L_2(q)$ onto $W_2^1(q)$.

Proof. Let $f \in W_2^1(q)$ and consider the equation in $L_2(q)$

(6) \[ f' = (TF + K_0)\phi. \]

We know that the index of $(TF+K_0)$ is 1. Thus $\alpha(TF+K_0) \geq 1$. Let $\phi_0 \in L_2(q)$ satisfy the equation

(7) \[ TF\phi_0 + K_0\phi_0 = 0. \]

Recall that

\[ K_0\phi_0 = \int_{-1}^{1} k(t, x)\phi(t) dt, \quad k(t, x) = h'(|t-x|) \sim (t-x) \log |t-x|. \]

Now $k(t, x)$ is Hölder continuous in $x$ uniformly in $t$ (see [4, p. 17]). Thus an easy argument shows that if $\phi_0 \in L_2(q)$, $K_0\phi_0$ is Hölder continuous. Thus applying the operator $F^{-1}T_0^{-1}$ we see that

\[ f' = (TF+K_0)\phi = (K_0\phi_0)' = C. \]

but from this we see that $\phi_0 \in h$. Hence all solutions of (7) in $L_2(q)$ are at the same time in $h$. Hence applying arguments as in [1] we see that there exists exactly 1 linearly independent solution of (6) in $L_2(q)$, say $\phi_0$. Further $L\phi_0 \equiv C_0$ where $C_0$ is a nonzero constant. Thus $\alpha(TF+K_0) = 1$, $\beta(TF+K_0) = 0$, i.e. $TF+K_0$ is onto. Let $\phi_f$ be a solution of (6). Then we consider the function $f - L\phi_f$. This is a function in $W_2^1(q)$ with derivative $f' - (TF+K_0)\phi_f = 0$ a.e. Thus $f - L\phi_f = C_f$ where $C_f$ is a definite constant. Thus $\phi^* = \phi_f + (C_f/C_0)\phi_0$ satisfies $L\phi^* = f$. The above argument shows that this solution is unique. \[ \square \]

Theorem 6. $L^{-1}$ is a continuous mapping from $W_2^1(q)$ onto $L_2(q)$.

Proof. Apply Banach's open mapping theorem.

Finally we note that if $f'$ is Hölder continuous and $\phi$ is the solution of $L\phi = f$ we have $(TF+K)\phi = f'$ and applying the operator $F^{-1}T_0^{-1}$ as is the proof of Theorem 5 we again see that $\phi \in h$. Hence the solutions found here coincide with those found in [1].

Addendum. It has recently come to my attention that similar results can be obtained if we consider the integral operator as mapping $L_p$ onto $W_p^1$ with $1 < p < 2$. Here $W_p^1$ is the usual Sobolev space. The key step (the analogue of Theorem 2) is supplied by Theorem 2 of [5].

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References


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