A CLASS OF UNIFORM CONVERGENCE STRUCTURES

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Abstract. In 1967, Cook and Fischer introduced in the journal Mathematische Annalen the notion of a uniform convergence structure, abbreviated u.c.s., for a set X. Here we consider the class Γ of u.c.s. which have the following property: a u.c.s. I ∈ Γ provided there is a filter Φ ∈ I such that Φ is finer than Φ(x) for every filter F which converges to x, for each x ∈ X. Various properties of the class Γ are discussed. The main result is that a topology τ for X is regular if and only if there is an I ∈ Γ such that I induces τ. Also it is shown that each I ∈ Γ induces a regular topology for X.

The class Γ₀ of u.c.s. which satisfy the completion axiom was first introduced by Biesterfeldt, Indag. Math., 1966. Here it is shown that Γ₀ ⊆ Γ and a characterization of the class Γ₀ is given in terms of Cauchy filters.


Let X be any set. Denote by Γ, the class of all uniform convergence structures, abbreviated u.c.s., for X with the following property: for each I ∈ Γ there is a Φ ∈ I such that τ_I(x) is the collection of all filters on X which are finer than Φ(x), for each x ∈ X.

Various properties of the class Γ are discussed. The main result is that any topology for X is regular if and only if there is a u.c.s. in Γ that induces the given topology.

Finally the u.c.s. for X which satisfy the completion axiom are discussed. The completion axiom was first introduced in [1].

2. A characterization for regular topologies. Let I ∈ Γ. Then there is a symmetric filter Φ ∈ I with Φ ≤ [Δ] (Φ coarser than the diagonal filter) such that τ_I(x) is the collection of all filters on X which are finer than Φ(x).

Proposition 1. Let I ∈ Γ. Then τ_I is a topology for X.

Proof. From [2], we must show that for A(x) ∈ Φ(x), A = A⁻¹ ∈ Φ, there is an H ∈ Φ(x) such that for each y ∈ H, A(x) ∈ Φ(y). Since Φ²(x) converges to x, we have that Φ²(x) = Φ(x). Hence there is a B ∈ Φ such that B²(x) ⊆ A(x). Let H = B(x) ∈ Φ(x) and let y ∈ H. We claim that A(x) ⊆ B(y) ∈ Φ(y). If z ∈ B(y), then (y, z) ∈ B. Since y ∈ B(x), then (x, y) ∈ B and it follows that (x, z) ∈ B² or z ∈ B²(x) ⊆ A(x).

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Hence the claim follows and thus $\tau_I$ is a topology for $X$.

The proof of the following proposition is similar to that in [2] for uniform spaces and will be deleted here.

**Proposition 2.** Let $I \in \Gamma$ and $A \subset X$, $B \subset X \times X$. Then $\text{Cl}(A) = \bigcap_{V \in \Phi} V(A)$ and $\text{Cl}(B) = \bigcap_{V \in \Phi} V \circ B \circ V$.

**Corollary.** If $I \in \Gamma$, then $\text{Cl}(\Phi) \supseteq \Phi^3$ and hence $\text{Cl}(\Phi) \in I$.

**Proposition 3.** If $I \in \Gamma$, then $\tau_I$ is a regular topology.

**Proof.** We claim that $x \times \text{Cl}(\Phi(x)) \supseteq \text{Cl}(\Phi) \in I$ ($x$ is a fixed ultra-filter); of course, it suffices to show that $\{x\} \times \text{Cl}(A(x)) \subset \text{Cl}(A)$ where $A \in \Phi$. If $y \in \text{Cl}(A(x))$ and $B \in \Phi$, then $(B(x) \times B(y)) \cap A \neq \emptyset$.

Hence $(x, y) \in \text{Cl}(A)$ and the claim follows. Therefore $\tau_I$ is a regular topology.

**Theorem.** Let $(X, \tau)$ be a topological space with $\eta(x)$ denoting the neighborhood filter at $x \in X$. Then $\tau$ is regular iff there is an $I \in \Gamma$ inducing $\tau$.

**Proof.** Let $I \in \Gamma$ such that $\tau_I = \tau$. Then by Proposition 3, $\tau$ is regular.

Conversely, assume that $\tau$ is a regular topology. Denote by $\Psi_0 = \bigvee_{x \in X} \{x \times \eta(z)\}$, $\Phi = \Psi_0 \setminus \Psi_0^{-1}$, $F(X \times X)$ the collection of all filters on $X \times X$, and $B = \{\Phi^n | n = 1, 2, \cdots \}$. Clearly $B$ is a base for a u.c.s. $I$ for $X$. That is, $I = \{\Psi \in F(X \times X) | \Psi \supseteq \Phi^n \text{ for some } n = 1, 2, \cdots \}$ is a u.c.s. for $X$.

We claim that $\tau_I = \tau$. The regularity of $\tau$ implies property (3) of Theorem 1 of [5]. Hence from property (2) of the same theorem, we have that $\Phi(x) = \eta(x)$ for each $x \in X$. Thus $x \times \eta(x) = x \times \Phi(x) \supseteq \Phi$ and hence $\tau(x) \subset \tau_I(x)$.

Conversely, if $\tau \in \tau_I(x)$, then $x \times \tau \supseteq \Phi^n$ for some positive integer $n$. Hence $\tau = (x \times \tau)(x) \supseteq \Phi^n(x)$. Thus we must show that $\Phi^n(x) \supseteq \eta(x)$. Assume $n \geq 2$. Let $N \in \eta(x)$ be open. Using the regularity of $\tau$, there exists open neighborhoods $N_i \in \eta(x)$ ($i = 1, 2, \cdots, n$) such that $x \in \bigcap_{i=1}^{N_1} \subset \text{Cl}(N_1) \subset N_2 \subset \text{Cl}(N_2) \subset \cdots \subset \text{Cl}(N_n) \subset N$. For each $z \in X$, define

$$N_z = \begin{cases} N_1 & \text{for } z \in N_1, \\ N_2 & \text{for } z \in \text{Cl}(N_1) - N_1, \\ N_{k+1} - \text{Cl}(N_k) & \text{for } z \in \text{Cl}(N_k) - N_k \quad (k = 2, 3, \cdots, n - 1), \\ N_{k+1} - \text{Cl}(N_k) & \text{for } z \in N_{k+1} - \text{Cl}(N_k) \quad (k = 1, 2, \cdots, n - 1), \\ N - \text{Cl}(N_{n-1}) & \text{for } z \in \text{Cl}(N_n) - N_n, \\ X - \text{Cl}(N_n) & \text{for } z \in X - \text{Cl}(N_n). \end{cases}$$
We claim that
\[
\left[ \left( \bigcup_{x \in X} \langle \{z\} \times N_z \rangle \right) \bigcup \left( \bigcup_{x \in X} \langle \{z\} \times N_z \rangle \right)^{-1} \right]^{n}(x) \subseteq N.
\]

Let \( y \in \text{L.H.S.} \), \( z_0 = x \), \( z_n = y \), and \( A \) equal the set in brackets. Hence \((z_{i-1}, z_i) \in A \) for some \( z_i \in X \) \((i = 1, 2, \ldots, n)\). By computation, one can show that \( z_i \in N_{i+1} \) \((i = 1, 2, \ldots, n-1)\) and \( y \in \text{Cl}(N) \subseteq N \). Therefore our claim follows and we have that \( \Phi^n(x) = \eta(x) \) for each natural number \( n \) and each \( x \in X \). Thus \( \tau_I = \tau \).

**Proposition 4.** If \( \tau \) is a compact Hausdorff topology for \( X \), then there is exactly one \( I \in \Gamma \) inducing \( \tau \).

**Proof.** From \([2]\) we have that \( I = \{ \Phi \in F(X \times X) \mid \Phi \geq \mathcal{U} \} \), where \( \mathcal{U} = \{ \text{all neighborhoods of } \Delta \} \), induces \( \tau \). Of course \( I \in \Gamma \). Hence if \( I_1 \in \Gamma \) and induces \( \tau \), then we want to show that \( I = I_1 \).

Let \( \Phi_1 \in I_1 \) be a symmetric filter such that \( \tau(x) \) is the collection of all filters on \( X \) which are finer than \( \Phi_1(x) \). We claim that \( \mathcal{U} \geq \Phi_1 \circ \Phi_1 \). Let \( A_1 = \Phi_1^{-1}(\Phi) \). Since \( A_1(x) \times A_1(x) \subseteq A_1 \circ A_1 \) for each \( x \in X \), we have that \( \bigcup_{x \in X} (A_1(x) \times A_1(x)) \subseteq \mathcal{U} \) and is contained in \( A_1 \circ A_1 \). Hence the claim follows and thus \( I \subseteq I_1 \).

Conversely, we claim that \( \Phi_1 \geq \mathcal{U} \). Suppose there is a \( V \in \mathcal{U} \) such that for all \( A_1 \in \Phi_1 \), \( A_1 \cap V \neq \emptyset \). Assume w.l.o.g. that \( V \) is an open neighborhood of \( \Delta \). The set \( \{ A_1 \cap V \mid A_1 \in \Phi_1 \} \) is a base for a filter \( \mathcal{L} \) on \( X \times X \). Since \( (X \times X, \tau \times \tau) \) is compact, \((x, y) \in \text{adh} (\mathcal{L}) \) for some \( x, y \in X \). Hence \((x, y) \in \text{Cl}(A_1 \cap V) \) for each \( A_1 \in \Phi_1 \). Thus \((x, y) \in \text{Cl}(V) = V \) and \((x, y) \in \text{Cl}(A_1) \) for each \( A_1 \in \Phi_1 \). Since \( \Delta \subseteq V \), \( x \neq y \). Also \( \text{Cl}(\Phi_1)(x) = \Phi_1(x) \) and we have that \( x \in \text{Cl}(\{y\}) \). This contradicts \( \tau \) being Hausdorff. Hence \( \Phi_1 \geq \mathcal{U} \) and thus \( \Phi_1 \in I \). Let \( \Psi \in I_1 \). Then of course \( \Psi \geq \Phi \). By an identical argument just given for \( \Phi_1 \), we have that \( \Psi \geq \Phi \) and thus \( I = I_1 \).

3. **Completion axiom.** The following definition is easily seen to be equivalent to that given in \([1]\). A u.c.s. \( I \) is said to satisfy the completion axiom, abbreviated c.a., provided there is a base for \( I \) consisting of symmetric filters coarser than the diagonal filter such that for each Cauchy filter \( \mathcal{F} \) on \( X \), \( \mathcal{F} \times \mathcal{F} \geq \Phi \) for every \( \Phi \) in the base.

Let \( I \) satisfy the c.a. with base \( B \).

**Proposition 5.** If \( I \) satisfies the c.a. and \( \Phi \in B \), then \( \tau_I(x) \) is the collection of all filters on \( X \) which are finer than \( \Phi(x) \).

**Proof.** Clearly \( \Phi(x) \in \tau_I(x) \). If \( \mathcal{F} \in \tau_I(x) \), then \( \mathcal{F} \times \mathcal{F} \subseteq \tau_I(x) \). Let \( A \in \Phi \). Since \( I \) satisfies the c.a., then \( (\mathcal{F} \times \{x\}) \times (\mathcal{F} \times \{x\}) \subseteq A \) for
some $F \in \mathfrak{F}$. Hence $F \subseteq A(x)$, which implies that $\mathfrak{F} \supseteq \Phi(x)$ and thus the proposition follows.

Let $\Gamma_0$ denote the collection of u.c.s. for $X$ which satisfy the c.a. From the above proposition $\Gamma_0 \subseteq \Gamma$. Hence each $I \in \Gamma_0$ induces on $X$ a regular topology.

Let $C_I$ denote the collection of all Cauchy filters on $X$ [3].

**Proposition 6.** If $I$ is any u.c.s. for $X$, then $I \in \Gamma_0$ iff $\bigwedge_{\mathfrak{F} \in \mathfrak{E}_f} (\mathfrak{F} \times \mathfrak{F}) \in I$.

**Proof.** Clearly the necessity follows. Conversely, if $\bigwedge_{\mathfrak{F} \in \mathfrak{E}_f} (\mathfrak{F} \times \mathfrak{F}) \in I$ then let $B = \{ \Phi \in I | \Phi = \Phi^{-1}, \bigwedge_{\mathfrak{F} \in \mathfrak{E}_f} (\mathfrak{F} \times \mathfrak{F}) \}$. Since $x \in C_I$ for each $x \in X$, then $\Phi \subseteq [\Delta]$ for each $\Phi \in B$. Clearly $B$ is a c.a. base for $I$.

I conjecture that each $I \in \Gamma_0$ induces a completely regular topology on $X$.

**References**


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