CONCERNING CONTINUOUS SELECTIONS

SAM B. NADLER, JR. AND L. E. WARD, JR.[1]

ABSTRACT. Necessary and sufficient conditions are given in order that certain types of partially ordered continua admit a continuous selection on the hyperspace of nonempty compact connected subsets. We establish that the class of arcwise connected compacta which admit continuous selections on their space of subcontinua is a proper subclass of the dendroids. This class is also shown to be larger than the class of metrizable generalized trees.

1. **Introduction.** In what follows a *continuum* is a nonempty compact connected Hausdorff space. If X is a topological space, then 2^X denotes the space of all nonempty closed subsets of X, with the Vietoris topology [**9**]. The subspace of closed and connected subsets of X is denoted $C(X)$. A *continuous selection* for a family $\alpha \subset 2^X$ is a continuous function $\sigma:\alpha \to X$ such that $\sigma(A) \in A$ for each $A \in \alpha$.

Recently Kuratowski, Nadler, and Young [**7**] have proved that if X is a metrizable continuum, then a continuous selection for 2^X exists if and only if X is an arc. If one seeks a continuous selection on $C(X)$, however, then it is known that this can be done if X is a tree [**2**]. In this note we will prove that the existence of a continuous selection for $C(X)$ is equivalent to the hereditary unicoherence of X, provided X is a member of a large class of metrizable continua which includes the Peano continua. Therefore, as a corollary it follows that the dendrites are precisely those Peano continua X for which $C(X)$ admits a continuous selection.

Our proof of these results employs a theorem of Kelley [**3**] which asserts that if X is a metrizable continuum then $C(X)$ is arcwise connected. The corresponding result for 2^X is due to Borsuk and Mazurkiewicz [**1**]. We note that M. M. McWaters [**8**] has recently extended these theorems to the case of nonmetrizable continua.

2. **A selection theorem.**

LEMMA 1. *If S^1 is the unit circle in the complex plane then $C(S^1)$ is a 2-cell.*

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PROOF. Let K be the cone over S^1 obtained by joining each point of S^1 to a point $(0, 0, 2\pi)$ in 3-space by a straight line segment. We define $h: C(S^1) \to K$ by letting $h(S^1) = (0, 0, 2\pi)$. If A is a (possibly degenerate) arc of S^1, let l be the arc length of A and let a_0 be the point of A which divides A into two subarcs of equal length. If J is the line segment in K which joins a_0 to $(0, 0, 2\pi)$, we define $h(A)$ to be the point on J whose z-coordinate is l. It is routine to verify that h is a homeomorphism of $C(S^1)$ onto K.

LEMMA 2. *If S^1 is the unit circle in the complex plane then $C(S^1)$ admits no continuous selection.*

PROOF. If $\sigma: C(S^1) \to S^1$ is a continuous selection then, by Lemma 1, σ is, in effect, a retraction of a 2-cell onto its boundary. It is well known that no such retraction exists.

Recall that a *dendroid* is a metrizable continuum which is arcwise connected and hereditarily unicoherent.

LEMMA 3. *If X is a metrizable continuum which admits a continuous selection for $C(X)$, then X is a dendroid.*

PROOF. By a result of Kelley [3], $C(X)$ is arcwise connected, and if there exists a continuous selection $\sigma: C(X) \to X$ then X is also arcwise connected. If X is not a dendroid then it contains subcontniua A and B such that $A \cap B$ is not connected. Let x and y be points which lie in distinct components of $A \cap B$. Since $A = \sigma(C(A))$ is arcwise connected, A contains an arc J_A whose endpoints are x and y. Similarly B contains an arc J_B whose endpoints are x and y. Since $J_A - J_B$ and $J_B - J_A$ are nonempty, it follows that $J_A \cup J_B$ contains a simple closed curve S and therefore $\sigma | C(S)$ is a continuous selection, contrary to Lemma 2.

If Γ is a partial order on a set X then $\Gamma x = \{y \in X : (y, x) \in \Gamma\}$. By a *directed space* we mean a topological space X with a closed partial order Γ such that the family $\{\Gamma x : x \in X\}$ has the finite intersection property, i.e., if $x_1, x_2, \cdots, x_n$ are elements of X then $\Gamma x_1 \cap \Gamma x_2 \cap \cdots \cap \Gamma x_n$ is nonempty. A *generalized tree* is a compact directed space in which each of the sets Γx is an arc and each subcontinuum has a zero. It follows from the results of [10] and [6] that a dendrite is a metrizable generalized tree and that a compact connected directed space is a generalized tree if and only if it is hereditarily unicoherent and each of the sets Γx is connected. Further, if X is a compact metrizable directed space in which each of the sets Γx is connected, then it follows from a result of R. J. Koch [5] or [11] that X is arcwise connected.

THEOREM 1. *Let X be a compact metrizable directed space and suppose that Γx is connected for each $x \in X$. Then $C(X)$ admits a continuous selection if and only if X is a generalized tree.*

PROOF. In view of Lemma 3 it suffices to show that if X is a generalized tree then $C(X)$ admits a continuous selection. If $A \in C(X)$ then A has a zero, $z(A)$. We will show that z is continuous. Let A_α be a net in $C(X)$ and suppose $A_\alpha \to A$. Since X is compact we may assume that $\lim z(A_\alpha)$ exists. Then $\lim z(A_\alpha) \in A$ and therefore $z(A) \in \Gamma(\lim z(A_\alpha))$. On the other hand there are elements $x_\alpha \in A_\alpha$ such that $\lim x_\alpha = z(A)$. Since $z(A_\alpha) \in \Gamma x_\alpha$ for each α and since Γ is closed, we infer that $\lim z(A_\alpha) \in \Gamma z(A)$. Therefore $\lim z(A_\alpha) = z(A)$ and the theorem is proved.

It has been proved by Virginia Walsh Knight [**4**] that every Peano continuum is a directed space in which the sets Γx are connected. Therefore the following corollary is immediate.

COROLLARY. *If X is a Peano continuum then $C(X)$ admits a continuous selection if and only if X is a dendrite.*

We note that there is a simple direct proof of this corollary. For if X is a Peano continuum and if $C(X)$ admits a continuous selection, then X contains no simple closed curve by Lemma 2. Thus X is a dendrite. The converse follows from [**2**].

3. **Two examples.** It is tempting to conjecture that among the arcwise connected compacta the existence of a continuous selection for the space of subcontinua is equivalent to the property of being a dendroid. Lemma 3 shows that if such a continuous selection exists then the compactum is a dendroid. However, the converse is not true. To see this let A_n be the line segment joining $(-1, 0)$ and $(1, 2^{-n})$ in the plane, for $n = 0, 1, 2, \cdots$, and let T be the line serment joining $(-1, 0)$ and $(1, 0)$. It is apparent that $D_1 = T \cup \bigcup_{n=0}^{\infty} \{A_n\}$ is a dendroid.

LEMMA 4. *If σ is a continuous selection on $C(D_1)$ then $\sigma(T) = (-1, 0)$.*

PROOF. For each $n = 0, 1, 2, \cdots$ let $\alpha(2^{-n}) = A_n$. Clearly $\alpha(2^{-n}) \to T$ and there is an arc $B_n \subset C(D_1)$ whose endpoints are $\alpha(2^{-n-1})$ and $\alpha(2^{-n})$ and such that each member of B_n contains either A_{n+1} or A_n. We extend α to each segment $[2^{-n-1}, 2^{-n}]$ as a homeomorphism onto B_n. Then for any sequence $t_n \to 0$ it is clear that $\alpha(t_n) \to T$, and hence if we let $\alpha(0) = T$ then α defines an arc in $C(D_1)$. Now suppose $\sigma(T) = (t, 0)$ with $t > -1$. Since $\sigma\alpha([0, 1])$ is locally connected, it follows that

$\sigma\alpha(2^{-n})=(t_n, 0)$ with $t_n>-1$, for sufficiently large n. But then $\sigma\alpha(2^{-n})\notin\alpha(2^{-n})$ which is contrary to the hypothesis that σ is a selection.

THEOREM 2. *There exists a dendroid D such that $C(D)$ admits no continuous selection.*

PROOF. Let $D=D_1\cup D_2$ where D_2 is the reflection of D_1 about the origin (see Figure 1). Suppose there exists a continuous selection $\sigma: C(D)\to D$. Then $\sigma_1=\sigma | C(D_1)$ and $\sigma_2=\sigma | C(D_2)$ are also continuous selections. By Lemma 4, $\sigma(T)=\sigma_1(T)=(-1, 0)$ and $\sigma(T)=\sigma_2(T)=(1, 0)$, which is impossible.

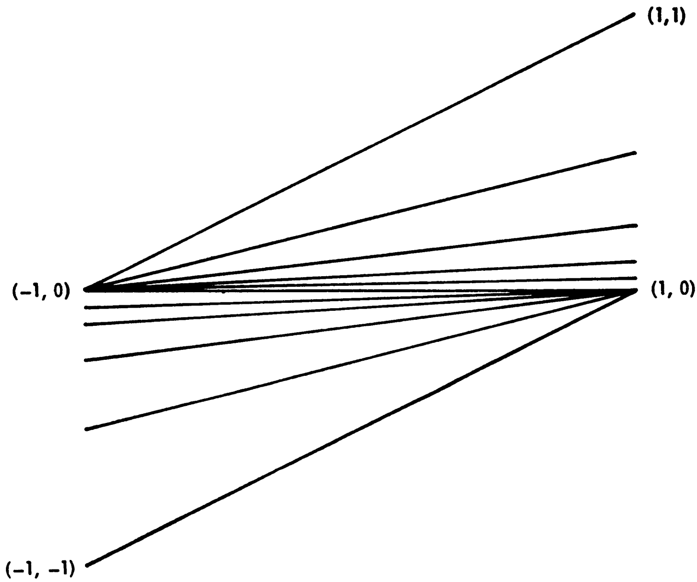

FIGURE 1

We have established that the class of arcwise connected compacta which admit continuous selections on the space of subcontinua is a proper subclass of the dendroids. The next example shows that this class is larger than the class of metrizable generalized trees. The example was first exhibited by Koch and Krule [6] who asserted that it is a dendroid which is not a generalized tree.

THEOREM 3. *There exists a dendroid D which cannot be partially ordered as a generalized tree and which admits a continuous selection $\sigma: C(D)\to D$.*

PROOF. In the plane let $A=\{(x, 2):0\leqq x\leqq 1\}$, let $T=\{(0, y):0\leqq y\leqq 2\}$, and for each $n=0, 1, 2, \cdots$ let $A_n=\{(2^{-n}, y):1\leqq y\leqq 2\}$. Let $D_1=A\cup T\cup\bigcup_{n=0}^{\infty}\{A_n\}$, let D_2 be the reflection of D_1 about the x-axis and let $D=D_1\cup D_2$ (see Figure 2).

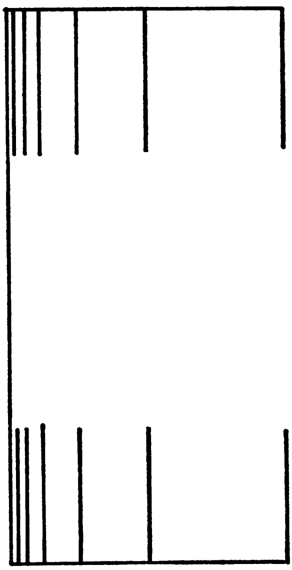

FIGURE 2

If D can be partially ordered as a generalized tree, with partial order $\leqq$, then D has a zero, 0. Without loss of generality we suppose $0\in D_2$. Then for each $n=0, 1, 2, \cdots$ we see that $(2^{-n}, 2)<(2^{-n}, 1)$ and that $(0, 1)<(0, 2)$, since for each $x\in D$ the set $\{y:y\leqq x\}$ must be an arc. However $(2^{-n}, 2)\rightarrow(0, 2)$ and $(2^{-n}, 1)\rightarrow(0, 1)$ so that if the partial order is closed then it must be that $(0, 2)\leqq(0, 1)$. Therefore D is not a generalized tree.

We shall now define a continuous selection on $C(D)$. Note first that D_1 is a generalized tree when $0=(0, 2)$, and also D_2 is a generalized tree when $0=(0, -2)$. Hence if $\sigma_i: C(D_i)\rightarrow D_i$ $(i=1, 2)$ is defined by $\sigma_i(A)=$ the zero of A relative to the natural partial order, then σ_i is a continuous selection. Moreover, by an argument similar to that of Lemma 4, $\sigma_1(T)=(0, 2)$ and $\sigma_2(T')=(0, -2)$, where T' is the reflection of T about the x-axis. For each $A\in C(T\cup T')$ let $\phi_1(A)$ be the second coordinate of $\sigma_1(A\cap T)$ if $A\cap T\neq\emptyset$, and otherwise let $\phi_1(A)=0$. Similarly, let $\phi_2(A)$ be the second coordinate of $\sigma_2(A\cap T')$ if $A\cap T'\neq\emptyset$, and otherwise let $\phi_2(A)=0$. Now define

$$\tilde{\sigma}(A) = (0, \phi_1(A) + \phi_2(A))$$

for $A \in C(T \cup T')$, and let $\tilde{\sigma} = \sigma_i$ on $C(D_i)$. Then $\tilde{\sigma}$ is a continuous selection on $C(D_1) \cup C(D_2) \cup C(T \cup T')$. The extension of $\tilde{\sigma}$ to a continuous selection σ on $C(D)$ is now straightforward. If $A \in C(D) - (C(D_1) \cup C(D_2) \cup C(T \cup T'))$ define

$$\sigma(A) = \tilde{\sigma}(A \cap (T \cup T')).$$

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State University of New York at Buffalo, Buffalo, New York 14226
Dalhousie University, Nova Scotia, Canada and
University of Oregon, Eugene, Oregon 97403