ON THE NUMBER OF RECURRENT ORBIT CLOSURES[1]

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ABSTRACT. Our purpose is to determine the maximum number of distinct recurrent orbit closures which can occur in a continuous flow on a compact nonorientable surface.

Let X be a surface and let $\boldsymbol{R}$ be the real numbers. By a *continuous flow* on X we mean a continuous mapping of $X \times \boldsymbol{R}$ to X such that for all x in X and for all s, t in $\boldsymbol{R}$ we have $x0 = x$ and $x(s+t) = (xs)t$ where xt denotes the image of (x, t). The *orbit* of x is defined by $\boldsymbol{O}(x) = \{xt : t \in \boldsymbol{R}\}$. We say x is *positively recurrent* if given any neighborhood U of x and any positive number T there exists $t > T$ such that $xt \in U$. Negative recurrence is defined analogously. The point x is *recurrent* if it is both negatively and positively recurrent. If x is positively or negatively recurrent, then $\mathrm{Cl}(\boldsymbol{O}(x))$ contains a recurrent point y such that $\mathrm{Cl}(\boldsymbol{O}(x)) = \mathrm{Cl}(\boldsymbol{O}(y))$. This follows from a simple category argument. Given a continuous flow on X we are interested in the number of distinct sets of the form $\mathrm{Cl}(\boldsymbol{O}(x))$ where x is recurrent and not periodic. Such a point will be called a strictly recurrent point.

A *local cross section* at y, an element of X, is a subset S of X containing y which is homeomorphic to a nondegenerate closed interval and for which there exists an $\epsilon > 0$ such that the map $(x, t) \to xt$ is a homeomorphism of $S \times [-\epsilon, \epsilon]$ onto the closure of an open neighborhood of y. If y is an interior point of X and if y is not a fixed point, then there exists a local cross section at y [**5**].

We will now state three theorems which are essential in the proof of our theorem. The first theorem shows us that we do not have to worry about descending chains of recurrent orbit closures. It was proved by Maĭer [**3**] for compact orientable surfaces, but one easily reduces the nonorientable case to the orientable one by lifting the flow to the orientable double covering.

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THEOREM 1. *If x and y are strictly recurrent points of a continuous flow on a compact surface and if y is in* $\mathrm{Cl}(O(x))$, *then* $\mathrm{Cl}(O(y)) = \mathrm{Cl}(O(x))$.

Note that this does not preclude the possibility that $y \notin \mathrm{Cl}(O(x))$ and $\mathrm{Cl}(O(x)) \cap \mathrm{Cl}(O(y)) \neq \emptyset$. By *recurrent orbit closure* we shall mean a set which is the closure of the orbit of some strictly recurrent point. Maĭer [3] also determined the maximum number of recurrent orbit closures for a continuous flow on a compact orientable surface with the following theorem:

THEOREM 2. *A continuous flow on an orientable surface of genus g has at most g distinct recurrent orbit closures. Moreover, this number is always achieved by some flow.*

Finally the author [4] has solved this problem for the Klein bottle with

THEOREM 3. *Every recurrent point of a continuous flow on a Klein bottle is periodic.*

THEOREM 4. *A continuous flow on a compact nonorientable surface of genus g has at most $[(g-1)/2]$ distinct recurrent orbit closures. (As usual $[x]$ denotes the greatest integer less than or equal to x.) Moreover, this number is always achieved by some flow.*

PROOF. Suppose X is a compact nonorientable surface whose genus g is greater than one. If X has a boundary, we can represent X as a subspace of a compact surface Y such that $\mathrm{Cl}(Y-X)$ consists of a finite number of disks. Since each boundary curve is an invariant set, we can extend any flow on X to the interiors of these disks without adding any strictly recurrent points. Thus we may assume X has no boundary.

For a given flow on X let $\{x_1, \cdots, x_k\}$ be a set of strictly recurrent points with distinct orbit closures and suppose $k > [(g-1)/2]$. Let $A_i = \mathrm{Cl}(O(x_i))$. By Theorem 1 $x_j \notin A_i$ when $i \neq j$.

Let $(\tilde{X}, p)$ be the orientable double covering of X. Its genus is $g-1$ because $\chi(\tilde{X}) = 2\chi(X)$ ($\chi(\ )$ denotes the Euler characteristic). The flow on X can be lifted to a flow on $\tilde{X}$ such that p is equivariant; i.e., $p(xt) = p(x)t$ [2]. Let $\{x_{i1}, x_{i2}\} = p^{-1}(x_i)$ and let $A_{ij} = \mathrm{Cl}(O(x_{ij}))$. It is easy to see that each x_{ij} is strictly recurrent [4]. Since $k > [(g-1)/2]$ implies $2k > g-1$, we have $A_{i1} = A_{i2}$ for some i by Theorem 2. We may as well assume that this i is 1.

Let S be a local cross section at x_{12} such that $S \cap A_{ij} = \emptyset$ for $i \geqq 2$ and $j = 1, 2$. Choose τ to be the smallest positive real number such that $x_{11}\tau \in S$ and choose σ to be the largest nonnegative real number such that $x_{12}\sigma \in S$ and $\sigma \in [0, \tau]$. Let $\tilde{\alpha}$ be the simple curve obtained from $\{x_{11}t : t \in [\sigma, \tau]\}$ and the piece of S joining $x_{11}\tau$ and $x_{12}\sigma$. Then $\alpha = p \circ \tilde{\alpha}$ is a simple one sided curve in X.

Let S' be a local cross section at x_2 such that $S' \cap \alpha = \emptyset$ and $S' \cap A_j = \emptyset$, $j \neq 2$. There exists σ' and τ', positive real numbers such that $x_2\sigma', x_2\tau' \in S'$ and $\{x_2 t : t \in [\sigma', \tau']\}$ with the piece of S' joining $x_2\sigma'$ and $x_2\tau'$ is a simple closed two sided curve on X. Call it β. Because x_2 is recurrent it is clear that β cannot divide X. We can modify the flow on X so that β belongs to the set of fixed points and if $O(x) \cap \beta = \emptyset$ the orbit of x is unchanged [1]. Alternatively we could have used the C^0 closing lemma [2]. Now we cut X along β and get a new surface Y and a flow on it with $k-1$ recurrent orbit closures. First observe that Y is nonorientable because $\beta \cap \alpha = \emptyset$, and Y has two boundary curves. Let g' denote the genus of Y. We have the following equations:

$$\chi(X) = \chi(Y), \quad \chi(X) = 2 - (0 + g), \quad \chi(Y) = 2 - (2 + g').$$

Solving for g' we get $g' = g - 2$, and thus

$$[(g' - 1)/2] = [(g - 1)/2] - 1 < k - 1.$$

If we view our construction of Y as the induction step, we can complete the proof by getting the induction started. Because $g' = g - 2$ we actually use two induction arguments—one for the even integers and the other for the odd integers. In other words, we must establish the result for $g = 1$ and $g = 2$. The first is a corollary of the Poincaré-Bendixson Theorem and the second is Theorem 3.

For the second part of the theorem we begin with a continuous flow on the torus with a nowhere dense recurrent orbit closure. Modify the flow so that there are two closed disks of fixed points in the complement of the recurrent orbit closure and throw away the interiors of these disks. Using n copies of this flow we construct in the obvious way a continuous flow with n distinct recurrent orbit closures on an orientable surface of genus n with two boundary curves. By attaching a cross cap of fixed points to one or both of the boundary curves we obtain an nonorientable surface of genus $2n+1$ or $2n+2$. Finally note that

$$[(2n + 1 - 1)/2] = n, \qquad \text{and} \qquad [(2n + 2 - 1)/2] = n.$$

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