A PRODUCT SPACE WITH THE FIXED POINT PROPERTY

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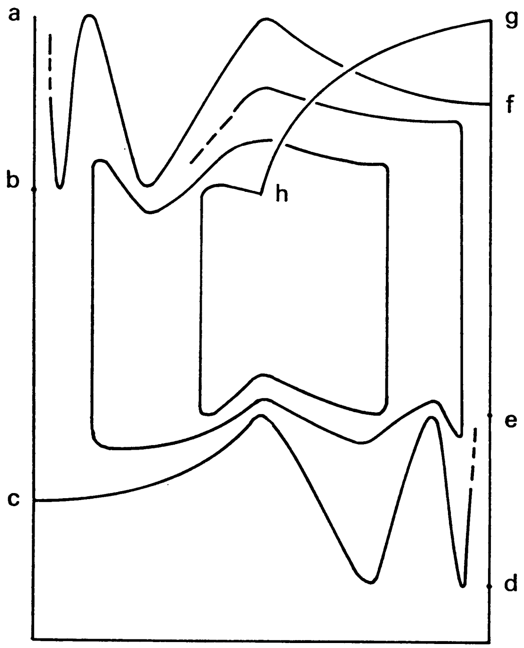
ABSTRACT. R. H. Bing, Amer. Math. Monthly **76** (1969), 119–132, utilizes an example of a 1-dimensional arcwise connected continuum X in E^3 with the fixed point property. His question (5) asks if $X \times I$ has the fixed point property. The answer is yes, and the proof given uses standard techniques of point-set topology.

A recent paper of Bing [1] contains a proof that the topological space X shown in the figure has the fixed point property (FPP), where X contains two sin $1/x$ curves $\hat{c}$ and $\hat{f}$ starting from c and f respectively with closures $\hat{c} \cup [d, e]$ and $\hat{f} \cup [a, b]$, and an "expanding spiral" $\hat{h}$ starting at h, with closure $\hat{h} \cup [a, c] \cup \hat{c} \cup [d, f] \cup \hat{f}$. All the



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indicated arcs and curves except $[g, h]$ are in a plane and only the end points of $[g, h]$ are in the plane. Question (5), [1], asks if $X \times I$ also has the FPP. The answer is yes and is given as the following proposition.

PROPOSITION. *$X \times I$ has the FPP.*

PROOF. This proof uses several suggestions made by the referee. Consider X as a subspace of ordinary 3-dimensional Euclidean space, and consider $X \times I$ as a metric space with the metric given by the sum of the metrics on X and I. Let $F: X \times I \rightarrow X \times I$ be a given continuous function and $\pi: X \times I \rightarrow X$ the projection. Denote by $\hat{c}_t$ any open infinite part, or tail, of $\hat{c}$ beginning at some point $x \in \hat{c}$, e.g. $\hat{c}_t(x) = \hat{c}_t = \hat{c} \sim [c, x]$, and similarly for $\hat{f}_t$ and $\hat{h}_t$.

Note, if Y is a locally connected compact space and $G: Y \rightarrow X \times I$ is a continuous function, then there exist tails $\hat{c}_t$, $\hat{f}_t$, and $\hat{h}_t$ all missing $\pi \circ G(Y)$, i.e. $\pi \circ G(Y) \cap (\hat{c}_t \cup \hat{f}_t \cup \hat{h}_t) = \emptyset$. This can be shown as follows: suppose every tail $\hat{f}_t$, say, intersects $\pi \circ G(Y)$. Then choose a sequence (z_i) in $\hat{f}$ converging to a point in $[a, b]$. Since $\hat{f}_t(z_i) \cap \pi \circ G(Y) \neq \emptyset$ for all i, there exists a sequence (y_i) in Y such that $\pi \circ G(y_i) \in \hat{f}_t(z_i)$ for all i. The compactness of Y implies the sequence (y_i) has an accumulation point $y \in Y$ and hence $\pi \circ G(y)$ is an accumulation point in X of the sequence $(\pi \circ G(y_i))$. Now, any point in $\hat{f}$ has a neighborhood intersection only a finite number of the tails $\hat{f}_t(z_i)$, and hence all accumulation points of the sequence $(\pi \circ G(y_i))$, are contained in $[a, b]$. Therefore $\pi \circ G(y) \in [a, b]$. Let U be a small neighborhood of $\pi \circ G(y)$ with the property that the closure $\overline{U}$ does not contain the point c and contains at most one of the points a and b. Then the component of U containing $\pi \circ G(y)$ is $[a, c] \cap U$. The local connectivity of Y implies there exists a connected neighborhood V of y such that $\pi \circ G(V) \subset U$. But this implies $\pi \circ G(V) \subset [a, c]$, and hence $[a, c]$ contains points of the sequence $(\pi \circ G(y_i))$. This is a contradiction, and so there must exist a tail $\hat{f}_t$ missing $\pi \circ G(Y)$. A similar argument shows there exist tails $\hat{c}_t$ and $\hat{h}_t$ missing $\pi \circ G(Y)$.

Now consider the following cases:

Case 1. Assume that either every tail $\hat{c}_t$ intersects $\pi \circ F(\hat{c} \times I)$ or every tail $\hat{f}_t$ intersects $\pi \circ F(\hat{f} \times I)$. Since both situations are handled identically, suppose the latter statement is true. The following shows this implies $F([a, b] \times I) \subset [a, b] \times I$. Choose a sequence (z_i) in $\hat{f}$ converging to a point in $[a, b]$. Since $\hat{f}_t(z_i) \cap \pi \circ F(\hat{f} \times I) \neq \emptyset$ for all i, there exists a sequence of points (x_i, u_i) in $\hat{f} \times I$ such that $\pi \circ F(x_i, u_i) \in \hat{f}_t(z_i)$ for all i. Let (x, u) be an accumulation point of the sequence (x_i, u_i). Then $\pi \circ F(x, u)$ is an accumulation point of the sequence

$(\pi \circ F(x_i, u_i))$. Note, any point in $\hat{f}$ has a neighborhood which intersects only a finite number of the tails $\hat{f}_t(z_i)$, so all accumulation points of the sequence $(\pi \circ F(x_i, u_i))$ are contained in $[a, b]$. Therefore $\pi \circ F(x, u) \in [a, b]$, i.e. $F(x, u) \in [a, b] \times I$. Also, the point (x, u) cannot be contained in $\hat{f} \times I$ since otherwise there exists a small neighborhood V of (x, u) such that $\overline{V}$ is locally connected and this would imply there exists a tail $\hat{f}_t$ missing $\pi \circ F(\overline{V})$. This is impossible since. $\pi \circ F(x, u)$ is an accumulation point of the sequence $(\pi \circ F(x_i, u_i))$ contained in $\hat{f}$. The only possibility is $(x, u) \in [a, b] \times I$ and hence $F([a, b] \times I) \cap ([a, b] \times I) \neq \emptyset$.

The following shows $\pi \circ F([a, b] \times I) \cap ([b, c] \sim \{b\}) = \emptyset$. Suppose not and choose a point $(y, v) \in [a, b] \times I$ such that $\pi \circ F(y, v) \in ([b, c] \sim \{b\})$. Since $\pi \circ F$ is uniformly continuous, there exists a real number $\delta > 0$ such that any two points in $X \times I$ within δ of each other are mapped onto points within, say, ϵ of each other where ϵ is some positive real number less than one half of each of the distances from the points $\pi \circ F(y, v)$ and f to the set $[a, b]$. Let $\alpha: I \to [a, b] \times I$ be a parametrization of the line segment in $[a, b] \times I$ with endpoints $(x, u) = \alpha(0)$ and $(y, v) = \alpha(1)$. From the sequence (x_i, u_i) chosen above, choose a point (x_j, u_j) within $\delta/2$ of (x, u). Now choose a point $(y_j, v_j) \in \hat{f} \times I$ within δ of (y, v) such that the projection in $\hat{f}$ of the shortest arc from (x_j, u_j) to (y_j, v_j) does not contain a neighborhood of an extreme point of the sin $1/x$ curve $\hat{f}$, i.e. no points of the projection other than possibly the endpoints are maximum or minimum values of the sin $1/x$ curve. Let $\alpha_j: I \to \hat{f} \times I$ be a parametrization of this shortest arc such that $\alpha_j(0) = (x_j, u_j)$, $\alpha_j(1) = (y_j, v_j)$, and for all $s \in I$, the points $\alpha(s)$ and $\alpha_j(s)$ are within δ of each other. Then $\pi \circ F \circ \alpha: I \to X$ is a path from $\pi \circ F(x, u) \in [a, b]$ to $\pi \circ F(y, v) \in [b, c] \sim \{b\}$, and $\pi \circ F \circ \alpha_j: I \to X$ is a path from $\pi \circ F(x_j, u_j) \in \hat{f}$ to $\pi \circ F(y_j, v_j)$. Note, the point $\pi \circ F \circ \alpha_j(1) = \pi \circ F(y_j, v_j)$ is within ϵ of the point $\pi \circ F(y, v) \in [b, c] \sim \{b\}$. This implies $\pi \circ F \circ \alpha_j(1) \notin \hat{f}$, and hence there exists $s \in I$ such that $\pi \circ F \circ \alpha_j(s) = f$. Let s' be the least such s. Let s'' be the least $s \in I$ such that $\pi \circ F \circ \alpha(s) = \pi \circ F \circ \alpha(1) = \pi \circ F(y, v)$. Now s' cannot be equal to or less than s'' since otherwise the point $\pi \circ F \circ \alpha_j(s') = f$ would be more than ϵ away from the point $\pi \circ F \circ \alpha(s') \in [a, c]$. Conversely, s'' cannot be less than s' since otherwise the point $\pi \circ F \circ \alpha(s'') = \pi \circ F(y, v)$ would be more than ϵ away from $\pi \circ F \circ \alpha_j(s'') \in \hat{f}$. Therefore the assumption that the image of $(y, v) \in [a, b] \times I$ under $\pi \circ F$ is contained in $[b, c] \sim \{b\}$ is false, and hence $\pi \circ F([a, b] \times I) \cap ([b, c] \sim \{b\}) = \emptyset$.

Since $[a, b] \times I$ is pathwise connected and $\pi \circ F([a, b] \times I) \cap [a, b] \neq \emptyset$, the preceding paragraph implies $\pi \circ F([a, b] \times I) \subset [a, b]$, i.e.

$F([a, b]\times I)\subset([a, b]\times I)$. Therefore F has a fixed point since $[a, b]\times I$ is a homeomorph of the closed unit disc.

Case 2. Assume that there exist tails $\hat{c}_t$, $\hat{f}_t$, and $\hat{h}_t$ missing $\pi\circ F(\hat{c}\times I)$, $\pi\circ F(\hat{f}\times I)$, and $\pi\circ F(\hat{h}\times I)$ respectively. Let c', f', and h' be the respective starting points of $\hat{c}_t$, $\hat{f}_t$, and $\hat{h}_t$, and let $X_1=X\sim(\hat{c}_t\cup\hat{f}_t\cup\hat{h}_t)$. Note, $X_1\times I$ is locally connected and compact, and hence by the comment preceding Case 1 above, there exist tails $\hat{c}_t'$, $\hat{f}_t'$, and $\hat{h}_t'$, all missing $\pi\circ F(X_1\times I)$. Consequently, the function $r: F(X_1\times I)\to X_1\times I$ given by $r(x, u)=(y, u)$, where y is x, c', f', or h' depending on whether x is in X_1, $\hat{c}_t$, $\hat{f}_t$, or $\hat{h}_t$, is continuous. Since $X_1\times I$ is a contractible polyhedron, the function $r\circ F_1: X_1\times I\to X_1\times I$, where $F_1=F|_{X_1\times I}$, has a fixed point $(x, u)\in X_1\times I$, i.e. $r\circ F_1(x, u)=(x, u)$.

Now suppose $r\circ F_1(x, u)\neq F_1(x, u)$. Then by the definition of r, $\pi\circ F_1(x, u)\in(\hat{c}_t\cup\hat{f}_t\cup\hat{h}_t)$ and $\pi\circ r\circ F_1(x, u)=c'$, f', or h'. But, if $\pi\circ F_1(x, u)\in\hat{c}_t$, then $(x, u)\notin\hat{c}\times I$ and $\pi\circ r\circ F_1(x, u)=c'$. This would imply $r\circ F_1(x, u)\neq(x, u)$. The same conclusion follows if $\pi\circ F_1(x, u)$ is contained in either $\hat{f}_t$ or $\hat{h}_t$. Therefore $r\circ F_1(x, u)=(x, u)$ implies $r\circ F_1(x, u)=F_1(x, u)$, and the two statements together imply $F_1(x, u)=F(x, u)=(x, u)$, i.e. F has a fixed point.

Case 3. Assume there exist tails $\hat{c}_t$ and $\hat{f}_t$ missing $\pi\circ F(\hat{c}\times I)$ and $\pi\circ F(\hat{f}\times I)$ respectively, but each tail $\hat{h}_t$ intersects $\pi\circ F(\hat{h}\times I)$. Note, if h' is any point contained in both $\hat{h}$ and $\pi\circ F(\hat{h}\times I)$, then the entire tail $\hat{h}_t=\hat{h}_t(h')$ is contained in $\pi\circ F(\hat{h}\times I)$ since the tail $\hat{h}_t(h'')$ starting at any point $h''\in\hat{h}_t(h')$ contains a point h''' in $\pi\circ F(\hat{h}\times I)$ by assumption and $\pi\circ F(\hat{h}\times I)$ is a path connected set. The only path from h''' to h' in X has to contain the point h''. Let $\hat{h}_t$ denote a tail which is contained entirely in $\pi\circ F(\hat{h}\times I)$.

The following shows each tail $\hat{f}_t$ must intersect $\pi\circ F(\hat{c}\times I)$. Let $X_2=[a, c]\cup\hat{c}\cup[d, f]\cup\hat{f}$ and suppose there exists a tail $\hat{f}_t'\subset\hat{f}_t$ missing $\pi\circ F(\hat{c}\times I)$. Since both $[a, c]\times I$ and $[d, f]\times I$ are locally connected and compact, there exists a third tail $\hat{f}_t''\subset\hat{f}_t'$ missing $\pi\circ F(([a, c]\cup[d, f])\times I)$. Together with the original assumption that $\hat{f}_t\cap\pi\circ F(\hat{f}\times I)=\varnothing$, this implies $\hat{f}_t''\cap\pi\circ F(X_2\times I)=\varnothing$. Now choose a sequence (z_i) contained in $\hat{h}_t$ which converges to a point $z\in\hat{f}_t''$. Since $\hat{h}_t\subset\pi\circ F(\hat{h}\times I)$, there exists a sequence (x_i, u_i) in $\hat{h}\times I$ such that $\pi\circ F(x_i, u_i)=z_i$ for all i. Let (x, u) be an accumulation point of the sequence (x_i, u_i). Then $\pi\circ F(x, u)=z$. Note, if $(x, u)\in\hat{h}\times I$, then (x, u) has a neighborhood U such that the closure $\overline{U}$ is a locally connected compact set. This would imply there exists some tail $\hat{h}_t'$ such that $\hat{h}_t'\cap\pi\circ F(\overline{U})=\varnothing$. This is impossible since $\pi\circ F(U)$ contains points of the sequence $(z_i)=(\pi\circ F(x_i, u_i))$. Therefore the point (x, u)

has to be contained in the set X_2. But this implies $\pi \circ F(X_2 \times I) \cap \hat{f}_t''$ $\neq \emptyset$ which contradicts the above conclusion that $\pi \circ F(X_2 \times I)$ $\cap \hat{f}_t'' = \emptyset$. Thus $\pi \circ F(\hat{c} \times I) \cap \hat{f}_t \neq \emptyset$ for all tails $\hat{f}_t$.

The above situation is analogous to the assumption in Case 1 that every tail $\hat{f}_t$ intersects $\pi \circ F(\hat{f} \times I)$. Hence, the same argument used in Case 1 implies $\pi \circ F([d, e] \times I) \subset [a, b]$. In particular $\pi \circ F(\{d\} \times I)$ $\subset [a, b]$.

Now let c' be the starting point of $\hat{c}_t$ and let $X_3 = [a, d] \cup [c, c']$. Define a function $r: F(X_3 \times I) \rightarrow X_3 \times I$ by $r(x, u) = (y, u)$, where y is x, c', or d depending on whether x is in X_3, $\hat{c}_t$, or $X \sim (X_3 \cup \hat{c})$. Since $X_3 \times I$ is locally connected and compact, the set $\pi \circ F(X_3 \times I)$ does not intersect every tail in $\hat{c}, \hat{f}$, and $\hat{h}$, and hence r is a continuous function. Since $X_3 \times I$ is a contractible polyhedron, the function $r \circ F_3: X_3 \times I \rightarrow X_3 \times I$, where $F_3 = F|_{X_3 \times I}$, has a fixed point. Note, if $r \circ F_3(x, u)$ $\neq F_3(x, u)$, then $\pi \circ F_3(x, u) \in X \sim X_3$. But $\pi \circ F_3(x, u) \in \hat{c}_t$ implies $(x, u) \notin \hat{c} \times I$ and $\pi \circ r \circ F_3(x, u) = c'$, and $\pi \circ F_3(x, u) \in X \sim (X_3 \cup \hat{c})$ implies $(x, u) \notin [d, e] \times I$ and $\pi \circ r \circ F_3(x, u) = d$. In either case $r \circ F_3(x, u) \neq F_3(x, u)$ implies $r \circ F_3(x, u) \neq (x, u)$. Therefore any fixed point of $r \circ F_3$ must be a fixed point of F.

Reference

1. R. H. Bing, *The elusive fixed point property*, Amer. Math. Monthly **76** (**1969**), 119–132. MR **38** #5201.

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