TAME BOUNDARY SETS OF CRUMPLED CUBES IN E^3

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ABSTRACT. If a crumpled cube K in E^3 is re-embedded by a homeomorphism h such that $h(K)$ is tame from Ext $h(K)$ and F is a tame closed subset of Bd K which either has no degenerate components or consists entirely of degenerate components, then $h(F)$ is tame.

Hosay [4] and Lininger [5] have independently shown that any crumpled cube in E^3 may be re-embedded in E^3 so that it is tame from its exterior. The question arises as to what properties of subsets of the boundary of the crumpled cube remain invariant under such a re-embedding. In particular if F is a tame closed subset of the boundary of a crumpled cube K in E^3 and h is a homeomorphism of K into E^3 such that $h(K)$ is tame from Ext $h(K)$, is $h(F)$ tame? If F has no degenerate components, then it follows from a recent result of Cannon [2] along with results of Lister [6] and Loveland [7] that $h(F)$ is tame. We show below that $h(F)$ is tame in case F consists entirely of degenerate components. For an arbitrary closed set F the question is open.

THEOREM. *If K is a crumpled cube in E^3, F is a tame closed* 0*-dimensional subset of* Bd K, *and h is a homeomorphism of K into E^3 such that $h(K)$ is tame from* Ext $h(K)$, *then $h(F)$ is tame.*

PROOF. By Bing's characterization of tame 0-dimensional sets in E^3 [1] there is a finite collection C_1 of mutually exclusive simple neighborhoods $\{N_{11}, N_{12}, \cdots, N_{1n(1)}\}$ of mesh less than 1 covering F. Let $\epsilon_1 = (1/4)D_1$ where $D_1 = \rho(F, E^3 - \bigcup_{i=1}^{n(1)} N_{1i})$ and using a lemma of Daverman [3] let h_1 be an ϵ_1-homeomorphism of K into E^3 and S_1 be a polyhedral 2-sphere homeomorphically within ϵ_1 of Bd $h_1(K)$ such that $h_1(K) \subset \text{Int } S_1$. Then C_1 covers $h_1(F)$, $\rho(h_1(p), p) < \epsilon_1$, for each $p \in F$, and it is clear from the method Daverman uses in the proof of his lemma that $h_1(F)$ is tame.

Now there is a finite collection C_2 of mutually exclusive simple neighborhoods $\{N_{21}, N_{22}, \cdots, N_{2n(2)}\}$ of mesh less than 1/2 such that C_2 covers $h_1(F)$ and each element of C_2 is a subset of an element of C_1. Let $\epsilon_2 = (1/4)D_2$ where $D_2 = \min\{\epsilon_1, \rho(h_1(F), E^3 - \bigcup_{i=1}^{n(2)} N_{2i})\}$,

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and let h_2' be an ϵ_2-homeomorphism of $h_1(K)$ into E^3 and S_2 be a polyhedral 2-sphere homeomorphically within ϵ_2 of Bd $h_2'(h_1(K))$ such that $h_2'(h_1(K))\subset \text{Int } S_2$. Then let $h_2=h_2'h_1$ and note that for each $p\in F$, $\rho(h_2(p),\ h_1(p))<\epsilon_2$, $\rho(h_2(p),\ p)<\epsilon_1+\epsilon_2\leqq\epsilon_1+(1/4)\epsilon_1\leqq(1/4)D_1+(1/16)D_1$, and $h_2(F)$ is tame.

Proceeding inductively we obtain

(1) a nested sequence of collections $C_1, C_2, \cdots, C_m, \cdots$ of mutually exclusive simple neighborhoods such that $C_m=\{N_{m1}, N_{m2}, \cdots, N_{mn(m)}\}$ is of mesh less than $1/2^{m-1}$ and covers $h_i(F)$ for $0<i<m$,

(2) a sequence of homeomorphisms $h_1, h_2, \cdots, h_m, \cdots$ of K into E^3 where $h_m=h_m'h_{m-1}$, h_m' is an ϵ_m-homeomorphism of $h_{m-1}(K)$ into E^3, $\epsilon_m=(1/4)D_m$, and $D_m=\min\{\epsilon_{m-1},\ \rho(h_{m-1}(F),\ E^3-\bigcup_{i=1}^{n(m)} N_{mi})\}$, and

(3) a sequence of polyhedral 2-spheres $S_1, S_2, \cdots, S_m, \cdots$ where S_m is homeomorphically within ϵ_m of Bd $h_m(K)$ and such that $h_m(K)\subset\text{Int } S_m$.

Note that for each $p\in F$, for $i>m$,

$$\rho(h_i(p),\ h_m(p)) < (1/4 + 1/16 + \cdots)D_m = (1/3)D_m.$$

Following Lininger [5] we obtain a limiting homeomorphism h' of the sequence $h_1, h_2, \cdots, h_m, \cdots$ such that $h'(K)$ is tame from Ext $h'(K)$. Then from the last inequality we have for any positive integer m, $h'(F)\subset\bigcup_{i=1}^{n(m)} N_{mi}$. Hence $h'(F)$ is tame and it follows easily that $h(F)$ is tame.

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